Physics 106b – Problem Set 8 – Due Jan 14, 2005

Version 2

January 13, 2005

These problems cover the material on central forces, gravitational orbits, and scattering in Hand and Finch Chapter 4 and Section 4 of the lecture notes. Please continue to write down roughly how much time you are spending on each problem.
Corrections:

- Problem 3: There is ambiguity in what is the correct polar formula to use. See the problem.

For the $1/r$ potential, the notes use $G\,\mu\,M$ as the constant coefficient; this is generalized to k in some of the problems below. That is, all the Keplerian formulae in the notes apply with the substitution $G\,\mu\,M \to k$.

Don't let the length scare you; the length is in the hints.

1. (a variant on Goldstein 3.18) A particle is in an elliptical gravitational orbit with semimajor axis a and eccentricity ϵ. The orbit is aligned as discussed in class, with the perigee (point of closes approach to the center of mass at the origin) along the $+x$ axis.

   (a) At the moment it reaches perigee, the particle experiences an impulse S in the *radial* direction. (Recall, the impulse S is the change in the momentum due to a force that is exerted for an infinitesimally short time.) The particle is thereby forced onto a new elliptical orbit. Which orbit parameters are left constant by the impulse? Which ones change? In addition to the orbit parameters we have explicitly discussed, consider the orbit orientation (the angle ϕ which obtains r_{min}.

   (b) Now suppose that the impulse at perigee is exactly *tangential* (perpendicular to the radial direction and along the velocity vector at perigee). Again, which orbit parameters are left constant and which ones change? Consider the orientation again also.

   You do not have to explicitly calculate all the new orbit parameters, but you must provide enough explanation to justify your choice of fixed and changing parameters. Can you think of an application of these results to designing course corrections for satellites?

2. (a variant on Hand and Finch 4.24/Goldstein 3.21) Show that the motion of a particle in the potential field

$$U(r) = -\frac{k}{r} + \frac{\beta}{r^2}$$

   results in a precessing orbit of the form

$$\frac{p}{r} = 1 + \epsilon \cos \alpha\phi$$

(Derive this result; *i.e.*, the goal is not to just check that the precessing form is correct, but to derive it.) Assuming the additional potential term is small compared to the Keplerian potential and the centrifugal energy term, show that the angular speed of precession of the orbit is

$$\dot{\Omega} = 2\pi \frac{\mu \beta}{l_\phi^2 \tau}$$

where τ is the period of the elliptical orbit.

3. (variant on Hand and Finch 4.30/Goldstein 3.13): Characteristics of a $F = -k/r^5$ potential:
   (a) Suppose a particle is subject to a central force $F(r) = -k/r^n$. Let the particle's orbit describe a circle of radius r_0 that passes through the center of force (the origin). Show that this implies $n = 5$.
   (b) Find the total energy E of this particular orbit.
   (c) Find the period of the orbit.
   (d) Find the velocity components $\dot{x}$ and $\dot{y}$ and the magnitude v as a function of the standard orbital angle ϕ and demonstrate that all three quantities become infinite when the particle passes through the origin.

   Hints:

   - You have two choices for which polar orbit equation to use. The first is

     $$r(\phi) = r_0 \sqrt{2\,(1 + \cos 2\phi)}$$

     The above equation is always positive, so when multiplied by $\cos\phi$ to find $x(\phi)$, one gets $x < 0$ for part of the orbit, which contradicts the problem. It works because the $x < 0$ part of the orbit is a mirror image of the $x > 0$ part.
     The second choice of polar orbit is

     $$r(\phi) = 2\, r_0\, \cos\phi$$

     This gives the correct $x(\phi)$ and $y(\phi)$ signs. But it allows $r < 0$, which is not, strictly speaking, allowed. You may use whichever form you prefer. The point of the problem is to find the right *technique* for proving $n = 5$, it won't matter which one you use.
   - You should find it necessary to calculate $u = 1/r$ and its derivatives $du/d\phi$ and $d^2u/d\phi^2$. It is a bit of heavy algebra to calculate them, so they are provided here. The two forms are completely equivalent (as one can show).
     First form(for $r(\phi) = r_0\sqrt{2\,(1 + \cos 2\phi)}$):

     $$\frac{du}{d\phi} = 2\, r_0^2 u^3 \sin 2\phi$$
     $$\frac{d^2u}{d\phi^2} = 2\, r_0^2 u^3\, (3 - \cos 2\phi)$$

     Second form (for $r(\phi) = 2\, r_0\, \cos\phi$):

     $$\frac{du}{d\phi} = (2\, r_0)^2\, u^3 \sin\phi\cos\phi$$
     $$\frac{d^2u}{d\phi^2} = (2\, r_0)^2\, u^3 \left(1 + \sin^2\phi\right)$$

     You are **not** required to calculate the derivatives explicitly, you may use these formulae.

4. (variant on Goldstein 3.32) A central force potential frequently encountered in nuclear physics is the rectangular well, defined by

$$\begin{aligned} V &= 0 \qquad r > a \\ &= -V_0 \qquad r \le a \end{aligned}$$

(a) Show that the scattering produced by such a potential in classical mechanics is identical with the refraction of light rays by a sphere of radius a and relative index of refraction

$$n = \sqrt{\frac{E + V_0}{E}}$$

(This equivalence demonstrates why it was possible to explain refraction phenomena by Huygen's waves and Newton's mechanical corpuscles.)

(b) Let the angles by which a particle's path is rotated when it enters the well be α_* and when it leaves the well be β_*. β_* is not the angle between the incoming and outgoing velocities; it is the angle between the outgoing velocity and the velocity while traversing the well. Find formulae relating α_* and β_* to the incoming particle impact parameter (and the various parameters of the problem, E, V_0, and a). Show that these two angles are identical. If you can demonstrate that $\alpha_* = \beta_*$ without finding separate formulae for the two, feel free to do so.

(c) What is the total cross section? Don't make this harder than it is.

Hint: Make use of the optical analogy for doing parts (b) and (c)!
Note: One could work from the results of part (b) to demonstrate that the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{n^2 a^2}{4\cos\frac{\theta_*}{2}} \frac{\left(n\cos\frac{\theta_*}{2} - 1\right)\left(n - \cos\frac{\theta_*}{2}\right)}{\left(1 + n^2 - 2\,n\cos\frac{\theta_*}{2}\right)^2}$$

(where $\theta_* = 2\alpha_*$ is the overall scattering angle), but you are **not** required to do so (pretty grungy algebra).