Physics 106a/196a – Final Exam – Due Dec 9, 2005

Instructions

Material: All lectures through Nov 29, Sections 1, 2, and 3 of lecture notes. (No special relativity!) Emphasis is on the second half of the term, material from Oct 20 onward. Only Ph196 students are responsible for the following topics:

- virtual work and generalized forces
- derivation of Euler-Lagrange equations via virtual work
- nonholonomic constraints
- Lagrangians for nonconservative forces
- Lagrange multipliers for nonholonomic constraints
- Legendre transformations
- All topics in "Theoretical Mechanics", Section 3.4 of lecture notes. Adiabatic invariance may be used on the exam for 106 students, but in a way that doesn't require them to have covered the material in Section 3.4 (á la PS 6-4).
- Congruence transformation, etc. subsection in Section 3.2

Review the material ahead of time, consult me, the TAs, your fellow students, or other texts if there is material you are having trouble with.

Logistics: The exam consists of this page plus 3 pages of exam questions. Do not look at the exam until you are ready to start it. Please use a blue book if possible (makes grading easier), but there will be no penalty if you don't have one. Problems 1 and 2 should be done by 106 students only, problems 6 and 7 by 196 students only. **All students should do any *two* of problems 3, 4, and 5.** A complete exam consists of **four** problems.

Time: 4 hrs, fixed time. You may take as many breaks as you like, but they may add up to no more than 30 minutes (2 x 15 minutes, 3 x 10 minutes, etc.).

Reference policy: Hand and Finch, Thornton (including solutions manual), official class lecture notes and errata, problem sets and solutions, your own lecture notes or other notes you have taken to help yourself understand the material. No other textbooks, no web searches, no interaction with your fellow students. Calculators and symbolic manipulation programs are neither needed nor allowed.

Due date: Friday, Dec 9, 5 pm, my office (311 Downs). 5 pm means 5 pm. Late exams will require extenuating circumstances; otherwise, no credit will be given.

Grading: Each problem is 25 points out of 100. The exam is 30% of the class grade.

Suggestions on taking the exam:

- Go through and figure out roughly how to do each problem first; make sure you've got the physical concept straight before you start writing down formulae.
- Don't fixate on a particular problem. They are not all of equal difficulty. Come back to ones you are having difficulty with.
- Don't get buried in algebra (this really should not be an issue on this exam). Get each problem to the point where you think you will get most of the points, then come back and worry about the algebra.

1. (106) (Courtesy of Roger Blandford, with minor embellishments) According to the French scholastic philosopher (and early dynamicist) Buridan, if an ass is placed midway between two bales of hay, it will not be able to decide between them and will starve. Suppose that the force of attraction of each hay-bale rises in direct proportion to the ass's distance from it (with constant of proportionality k). Write down the Lagrangian for the ass (which is unusual and exhibits no friction) and show that the point midway between the two bales is a stable equilibrium and that the motion is simple harmonic about that equilibrium point. The unfortunate ass will slowly waste away as it undergoes its periodic motion. Use the principle of adiabatic invariance and an argument by analogy to PS 6-4 (H&F 3-12) to show that, if the ass begins by oscillating between points one-quarter and three-quarters of the way between the bales, it will be able to satisfy its growing hunger only after it has declined to 1/16 of its original mass. (Incidentally, Buridan was tied in a sack and thrown in the Seine as a reward for his intellectual explorations.)

2. (106) A long string is terminated at its right end by a massless ring which slides on a vertical rod and is impeded by a frictional force proportional to its velocity. Define the x axis to be positive to the right and let the rod be at $x = 0$. Consider a right-going wave incident on the ring from the left. You will analyze the motion of the ring and the reflected wave.

   (a) Explain why the appropriate boundary condition at the ring is

   $$\tau \left.\frac{\partial y}{\partial x}\right|_{x=0} + b \left.\frac{\partial y}{\partial t}\right|_{x=0} = 0$$

   where τ is the string tension and b is the constant of proportionality of the frictional force. (Note that you may obtain different signs if you define your coordinate system orientation differently.)

   (b) Assume the incoming wave has amplitude coefficient 1. Show that the amplitude coefficient of the reflected wave is

   $$a_< = \frac{\tau - b\,v}{\tau + b\,v}$$

   where v is the propagation speed $\sqrt{\tau/\Lambda}$ on the string.

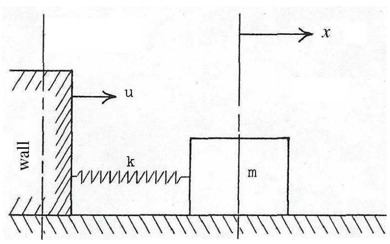
   (c) Calculate the reflected power fraction.

   (d) How does the reflected wave behave in the limiting cases of very large and very small friction? For what value of the friction constant is there no reflected wave?

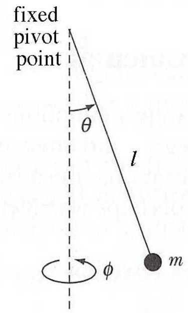
3. (106/196) A mass m is connected by means of a linear spring k to a wall which is suddenly accelerated from rest to speed u at $t = 0$, as shown in the figure. The mass is subject to a frictional force $F_f = -b\,\dot{x}$, with b small enough that the motion is underdamped (but don't neglect the damping!). The mass is at rest at $t = 0$ and the spring is initially at its rest length l. Define the x origin to be the position of the mass at $t = 0$. Do the following:

   (a) Show that the steady-state (asymptotic) behavior is $x(t \to \infty) = u\,t - \frac{b}{k}\,u$

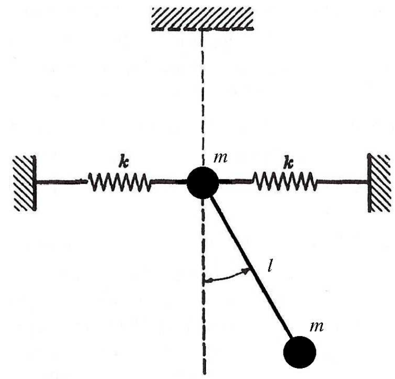
   (b) Determine the full behavior of the system as a function of time. You will find it much easier to do this by guessing (with appropriate motivational reasoning) the correct form for the solution using the above steady-state behavior than by using the Green's function, but you may resort to the Green's function if necessary. You may leave your solution in dimensionless form if that is convenient.



Problem 3



Problem 4



Problem 5

4. (106/196) A spherical pendulum consists of a mass m fixed by an inextensible string of length l to a pivot, subject to gravity, and allowed to move in all three dimensions. (See the figure.) The bob is allowed to go above the pivot point. Under some conditions, the string tension will go to zero and the string will collapse. For example, if the initial condition has the bob at rest a height l directly above the pivot, obviously the string will collapse and the bob will just fall down. Using spherical coordinates, find the equations of motion in r, θ, and ϕ, including a Lagrange multiplier to enforce the string length constraint. Using the Lagrange multiplier, obtain the tension in the string in terms of the polar angle θ and the initial energy E. Find the angle θ_1 at which the string will collapse in terms of E. You do not need to know the initial polar angle θ_0 or initial ϕ angular momentum p_ϕ, and you do not need to solve the equation of motion in θ to determine the tension.

5. (106/196) Consider the coupled oscillator system in the figure. There is a mass m acted on by two springs, each with spring constant k, and **constrained to move in x only**. Suspended from the mass m via a massless rod of length l is a pendulum with bob mass m. Find the normal mode frequencies and vectors (don't bother to normalize them) of the system. Note that this problem is like PS3-3, but now there is an additional mass at the point where the springs meet the pendulum pivot point.

6. (196) Two parts:

   (a) A ball bounces between two hard walls separated by a distance d. In between the walls, no force acts on the ball. Draw a phase trajectory for one complete cycle. Notice that the momentum changes direction discontinuously when the ball bounces from the wall. In dimensionless units, the energy is $E = p^2/2$. Calculate the action variable and the area in phase space for one cycle. Then calculate the frequency by differentiating the energy with respect to the action variable. Does this agree with an elementary calculation of the frequency $\omega = 2\pi/T$, with T being the period of the motion? It is not possible to define a Hamiltonian in this case because the force is discontinuous, but the action variable can still be defined. Is this also true of the angle variable?

   (b) You will now obtain the above as a limiting case of an analytic problem. A particle with $m = 1$ moves in a potential of the form

$$V(q) = U \tan^2(aq)$$

where U and a are positive constants. Find the turning points of the motion. Prove that the action variable I obeys the relation

$$\frac{a\,I}{\sqrt{2}} = \sqrt{E+U} - \sqrt{U}$$

where E is the total energy, and thus prove that the frequency ω has the energy dependence

$$\frac{\omega}{a\sqrt{2}} = \sqrt{E+U}$$

The increase of the frequency with energy reflects the fact that the restoring force increases with displacement q faster than linearly. Finally, explain what limiting values must be taken for a and U in order to obtain the setup described in part (a) and show that the limiting behaviors of $I(E)$ and $\omega(E)$ agree with part (a).

In doing this problem, you will encounter a difficult integral. You may use the formula

$$\int_{q_-}^{q_+} dq\,\sqrt{E - U\,\tan^2 aq} = \frac{\sqrt{U}}{a}\left[\sqrt{1+\frac{E}{U}}\,\arctan\left(\sqrt{1+\frac{E}{U}}\,\tan\theta\right) - \theta\right]\Bigg|_{\theta_-}^{\theta_+}$$

where $\sin\theta = \sqrt{\frac{U}{E}}\,\tan a\,q$ and $\theta_\pm$ correspond to the turning points $q_\pm$. (Why are we assured this change of variables is valid; *i.e.*, why is the right side never larger than 1 in magnitude?) If you are curious, this result can be obtained by

- Making the trigonometric substitution $\sin\theta = \sqrt{\frac{U}{E}}\,\tan a\,q$
- Use the following formula (from the *CRC Standard Math Tables*, a good investment for any physicist):

$$\int dx \frac{\cos^2 c\,x}{a^2 + b^2\,\sin^2\,cx} = \frac{\sqrt{a^2+b^2}}{a\,b^2\,c}\,\arctan\frac{\sqrt{a^2+b^2}\,\tan c\,x}{a} - \frac{x}{b^2}$$

7. (196) A system of n particles moves in a plane under the influence of interaction forces derived from potential terms depending only upon the scalar distances between particles.

   (a) Using plane polar (cylindrical) coordinates for each particle relative to a common origin, write down the form of the Hamiltonian for the system (the common origin is not necessarily at the center of mass).

   (b) Find a generating function for the canonical transformation that corresponds to a transformation to coordinates rotating in the plane counterclockwise with a uniform angular rate ω (the same for all particles). To be clear about the sign, if the particles were all orbiting around the origin at angular frequency ω, then in the new coordinate system they would be at rest. What are the transformation equations for the momenta?

   (c) What is the new Hamiltonian? What physical significance can you give to the difference between the old and new Hamiltonians?