Physics 106a/196a – Problem Set 6 – Due Nov 17, 2006

Version 2, November 16, 2006

This problem set focuses primarily on the simple harmonic oscillator, Section 3.1 of the lecture notes. It also includes a bit of material on action-angle variables and adiabatic invariance, Section 2.4.4 of the lecture notes. We covered this last topic in class on November 7. If you missed it, you'll need to skim through Section 2.4.4, but don't worry about the derivations (which use material on canonical transformations that was covered in the optional lectures during the week of Oct 30), just focus on the results. Examples 2.14, 2.15 and in Hand and Finch Section 6.5 will be instructive. For Ph196a students, there is more material on the Hamilton-Jacobi equation (Section 2.4.5 of the lecture notes). Problem 1 is for 106a students only, 2-4 are for 106a and 196a students, and 5 is for 196a students only.

Version 2: Clarify the "Why" question in Problem 1(b). Resolve ambiguity by replacing "damped" by "underdampled" in Problems 2 and 4. Explicitly say what the ϕ coordinate is in Problem 3.

1. (106a) Consider a much simplified model of a gas, which consists of a single free particle of mass m moving in one dimension between walls located at $x = 0$ and $x = L$. The walls can be treated as infinitely high potential barriers.

   (a) Calculate the action variable as a function of L and the energy E of the particle. Verify that the period of the motion is given by $T = 2\pi / (dE/dI)$.

   (b) If the particle moves very fast, it makes sense to speak of the "average force" $\langle F \rangle$ exerted on the walls by the particle. Express $\langle F \rangle$ as a function of L and E. Use adiabatic invariance to relate the rate of change of $\langle F \rangle$ and E to the rate of change of L. Why do the results for $\langle F \rangle$ and E differ in the way they do (consider the fractional rates of change $\frac{1}{\langle F \rangle}\frac{d\langle F \rangle}{dt}$ and $\frac{1}{E}\frac{dE}{dt}$)? What well known law have you demonstrated?

2. (106a/196a) A force $F_0 \cos(\omega' t + \phi_0)$ acts on an *underdamped* SHO beginning at $t = 0$. The damped SHO's natural frequency in the absence of damping is ω and in the presence of damping is ω'.

   (a) What must be the inital values of x and v in order that there be no transient term in the full solution?

   (b) If instead $x_0 = v_0 = 0$, find the transient term. What are its amplitude and phase in terms of F_0, ϕ_0, and the parameters of the damped SHO?

3. (106a/196a) Consider the hoop in Figure 1.11 of Hand and Finch. In addition, to the r and θ coordinates, let ϕ indicate the spherical ϕ azimuthal coordinate, giving the angle of rotation about the Z axis in the diagram (not about the hoop's axis!). The diagram indicates that the hoop (and thus the bead) advance in ϕ at rate Ω, $\phi = \phi_0 + \Omega\, t$. Two parts:

   (a) Write down constraints in r and ϕ, obtain the equations of motions in r, θ, ϕ with Lagrange multipliers, and write down the generalized constraint forces in the r and ϕ

coordinates as functions of θ, $\dot{\theta}$, and the fixed parameters of the problem. Explain the terms contributing to N_r. Why does N_ϕ depend on $\dot{\theta}$?

(b) Now, neglect the r and ϕ equations of motion and consider only the θ equation of motion. (If you are not sure you got the right EOM in (a) with Lagrange multipliers, start over and obtain the EOM with θ as the only dynamical coordinate.) Do the following:

i. Find the equilibrium value(s) of θ. Does the existence of any of them depend on Ω, the angular speed at which the hoop rotates?

ii. Determine the stability of the equilibrium points and, for the stable ones, the natural frequency of small-amplitude simple harmonic oscillation in θ about them.

iii. Suppose the bead is subject to friction as it slides along the hoop, with $F_f = -b\,v$ where v is the linear velocity (*i.e.*, m/s) along the hoop. What is the Q for the various stable oscillation possibilities in the presence of this damping term?

4. (106a/196a) An *underdamped* SHO is driven with the force $F(t) = F_0\left(1 - e^{-\alpha t}\right)$ starting at $t = 0$. The oscillator is at rest for $t < 0$. Find the full solution for the oscillator's response using the Green's function for the damped SHO. Yes, this gets to be a bit painful, be careful to not lose any terms and to make sure your answer makes sense.

5. (196a) For a conservative system, show that, by solving an appropriate partial differential equation, we can construct a canonical transformation such that the Hamiltonian is a function of the new *coordinates* only. (Do not use the exchange transformation $Q = p$, $P = -q$.) Show how a formal solution to the motion of the system is given in terms of the new coordinates and momenta, and how the initial conditions are applied.