Now, let us introduce the idea of an inner product, which lets us discuss normalization and orthogonality of vectors.

An inner product is a function that obtains a single complex number from a pair of vectors $|v\rangle$ and $|w\rangle$, is denoted by $\langle v|w \rangle$, and has the following properties:

- **positive definiteness**: $\langle v|v \rangle \geq 0$ with $\langle v|v \rangle = 0$ only if $|v\rangle = |0\rangle$; i.e., the inner product of any vector with itself is positive unless the vector is the null vector.
- **transpose property**: $\langle v|w \rangle = \langle w|v \rangle^*$, or changing the order results in complex conjugation.
- **linearity**: $\langle u|\alpha v + \beta w \rangle = \alpha \langle u|v \rangle + \beta \langle u|w \rangle$

This definition is specific to the case of vector spaces for which the field is the real or complex numbers. Technical problems arise when considering more general fields, and we will only use vector spaces with real or complex fields, so this restriction is not problematic.
Some notes:

▶ It is not necessary to assume $\langle v | v \rangle$ is real; the transpose property implies it.

▶ The above also implies antilinearity, $\langle \alpha v + \beta w | u \rangle = \alpha^* \langle v | u \rangle + \beta^* \langle w | u \rangle$

▶ Inner products are representation-independent — the above definitions refer only to the vectors and say nothing about representations. Therefore, if one has two representations of a linear vector space and one wants them to become representations of the same inner product space, the inner product must be defined consistently between the two representations.
Now, for some statements of the obvious:

An inner product space is a vector space for which an inner product function is defined.

The length or norm or normalization of a vector $\langle v \rangle$ is simply $\sqrt{\langle v | v \rangle}$, which we write as $|v|$. A vector is normalized if its norm is 1; such a vector is termed a unit vector. Note that a unit vector can be along any direction; for example, in $\mathbb{R}^3$, you usually think of the unit vectors as being only the vectors of norm 1 along the $x$, $y$, and $z$ axes; but, according to our definition, one can have a unit vector along any direction.

The inner product $\langle v | w \rangle$ is sometimes called the projection of $\langle w \rangle$ onto $\langle v \rangle$ or vice versa. This derives from the fact that, for $\mathbb{R}^3$, the inner product reduces to

$$\langle v | w \rangle = |v| |w| \cos \theta_{vw}$$

where $\theta_{vw}$ is the angle between the two vectors. In more abstract spaces, it may not be possible to define an angle, but we keep in our minds the intuitive picture from $\mathbb{R}^3$. In general, the two vectors must be normalized in order for this projection to be a meaningful number: when you calculate the projection of a normalized vector onto another normalized vector, the projection is a number whose magnitude is less than or equal to 1 and tells what (quadrature) fraction of $\langle w \rangle$ lies along $\langle v \rangle$ and vice versa. We will discuss projection operators shortly, which make use of this definition. Note that the term “projection” is not always used in a rigorous fashion, so the context of any discussion of projections is important.
Two vectors are **orthogonal** or **perpendicular** if their inner product vanishes. This is equivalent to saying that their projections onto each other vanish.

A set of vectors is **orthonormal** if they are mutually orthogonal and are each normalized; i.e., $\langle v_i | v_j \rangle = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker delta symbol, taking on value 1 if $i = j$ and 0 otherwise. We will frequently use the symbol $|i\rangle$ for a member of a set of orthonormalized vectors simply to make the orthonormality easy to remember.
Calculating Inner Products

As usual, the above definitions do not tell us algorithmically how to calculate the inner product in any given vector space. The simplest way to do this is to provide the inner products for all pairs of vectors in a particular basis, consistent with the rules defining an inner product space, and to assume linearity and antilinearity. Since all other vectors can be expanded in terms of the basis vectors, the assumptions of linearity and antilinearity make it straightforward to calculate the inner product of any two vectors.

That is, if \( \{ |j\rangle \} \) is a basis (not necessarily orthonormal), and \( |v\rangle = \sum_{j=1}^{n} v_j |j\rangle \) and \( |w\rangle = \sum_{j=1}^{n} w_j |j\rangle \), then

\[
\langle v | w \rangle = \left\langle \sum_{j=1}^{N} v_j (j) \right| \sum_{k=1}^{N} w_k (k) \right\\
\]

where \((j)\) and \((k)\) are the \(|j\rangle\) and \(|k\rangle\) vectors. Using linearity and antilinearity,

\[
\langle v | w \rangle = \sum_{j=1}^{N} v_j^* \left\langle j \right| \sum_{k=1}^{N} w_k (k) \right\rangle = \sum_{j=1}^{N} \sum_{k=1}^{N} v_j^* w_k \langle j | k \rangle = \sum_{j,k=1}^{N} v_j^* w_k \langle j | k \rangle \quad (3.4)
\]

Once we know all the \(\langle j | k \rangle\) inner products, we can calculate the inner product of any two vectors.
Of course, if the basis is orthonormal, this reduces to

$$\langle \nu | w \rangle = \sum_{jk} v^*_j w_k \delta_{jk} = \sum_j v^*_j w_j \quad (3.5)$$

and, for an inner product space defined such that component values can only be real numbers, such as $\mathbb{R}^3$ space, we just have the standard dot product. (Note that we drop the full details of the indexing of $j$ and $k$ when it is clear from context.)

With the assumptions that the basis elements satisfy the inner product space rules and of linearity and antilinearity, the transpose property follows trivially. Positive definiteness follows nontrivially from these assumptions for the generic case, trivially for an orthonormal basis.

Note also that there is no issue of representations here — the inner products $\langle j | k \rangle$ must be defined in a representation-independent way, and the expansion coefficients are representation-independent, so the inner product of any two vectors remains representation-independent as we said it must.
Example 3.10: $\mathbb{R}^N$ and $\mathbb{C}^N$

The inner product for $\mathbb{R}^N$ is the dot product you are familiar with, which happens because the basis in terms of which we first define $\mathbb{R}^N$ is an orthonormal one. The same statement holds for $\mathbb{C}^N$, too, with the complex conjugation of the first member’s expansion coefficients. So, explicitly, given two vectors (in $\mathbb{R}^N$ or $\mathbb{C}^N$)

$$\left| \mathbf{v} \right-arrow \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}, \quad \left| \mathbf{w} \right-arrow \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

(note the use of arrows instead of equality signs to indicate representation!) their inner product is

$$\langle \mathbf{v} | \mathbf{w} \rangle = \sum_j v_j^* w_j$$

(Note the equality sign for the inner product, in contrast to the arrows relating the vectors to their representations — again, inner products are representation-independent.) Because the basis is orthonormal, the entire space is guaranteed to satisfy the inner product rules and the spaces are inner product spaces.
Example 3.11: Spin-1/2 particle at the origin

The three bases we gave earlier,

\[
\begin{align*}
|\uparrow_x\rangle & \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & |\downarrow_x\rangle & \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
|\uparrow_y\rangle & \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} & |\downarrow_y\rangle & \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \\
|\uparrow_z\rangle & \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} & |\downarrow_z\rangle & \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{align*}
\]

are each clearly orthonormal by the algorithm for calculating the \(\mathbb{C}^2\) inner product; e.g.,

\[
\langle \uparrow_y | \uparrow_y \rangle = \frac{[1 \cdot 1 + (-i) \cdot i]}{2} = 1 \quad \langle \uparrow_y | \downarrow_y \rangle = \frac{[1 \cdot 1 + (-i) \cdot (-i)]}{2} = 0
\]

(Note the complex conjugation of the first element of the inner product!) Hence, according to our earlier argument, the space is an inner product space.
It is physically interesting to explore the inner products between members of different bases. Some of them are

\[
\langle \uparrow_x | \uparrow_z \rangle = \frac{1}{\sqrt{2}} \quad \langle \uparrow_x | \downarrow_z \rangle = \frac{1}{\sqrt{2}} \quad \langle \downarrow_y | \uparrow_x \rangle = \frac{1 + i}{2} \quad \langle \downarrow_y | \downarrow_x \rangle = \frac{1 - i}{2}
\]

The nonzero values of the various cross-basis inner products again hint at how definite spin along one direction does not correspond to definite spin along others; e.g., \(|\uparrow_x\rangle\) has a nonzero projection onto both \(|\uparrow_z\rangle\) and \(|\downarrow_z\rangle\).
Example 3.12: The set of all complex-valued functions on a set of discrete points \(i \frac{L}{n+1}, i = 1, \ldots, n\), in the interval \((0, L)\), as in Example 3.4

We know that this is just a different representation of \(\mathbb{C}^N\), but writing out the inner product in terms of functions will be an important lead-in to inner products of QM states on the interval \([0, L]\). Our representation here is

\[
|f\rangle \leftrightarrow \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} \quad \text{with} \quad x_j = j \frac{L}{N+1} \equiv j \Delta
\]

We use the same orthonormal basis as we do for our usual representation of \(\mathbb{C}^N\),

\[
|1\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad |2\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad |N\rangle \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
\]

so that \(\langle j | k \rangle = \delta_{jk}\).
The inner product of two arbitrary vectors in the space is then

\[ \langle f | g \rangle = \sum_j f^*(x_j) g(x_j) \quad (3.6) \]

That is, one multiplies the conjugate of the first function against the second function point-by-point over the interval and sums. The norm of a given vector is

\[ \langle f | f \rangle = \sum_j f^*(x_j) f(x_j) = \sum_j |f(x_j)|^2 \quad (3.7) \]

We shall see later how these go over to integrals in the limit \( \Delta \to 0 \).
Example 3.13: The set of real, antisymmetric $N \times N$ matrices (with the real numbers as the field) with conjugation, element-by-element multiplication, and summing as the inner product (c.f., Example 3.5).

Explicitly, the inner product of two elements $|A\rangle$ and $|B\rangle$ is

$$\langle A | B \rangle = \sum_{jk} A^*_j B_k$$  \hspace{1cm} (3.8)$$

where $jk$ indicates the element in the $j$th row and $k$th column. We include the complex conjugation for the sake of generality, though in this specific example it is irrelevant. Does this inner product satisfy the desired properties?

- Positive definiteness: yes, because the inner product squares away any negative signs, resulting in a positive sum unless all elements vanish.
- Transpose: yes, because the matrix elements are real and real multiplication is commutative.
- Linearity: yes, because the expression is linear in $B_{kl}$.
This inner product makes this space a representation of $\mathbb{R}^{N(N-1)/2}$ as an inner product space. Let's write down normalized versions of the bases we considered previously:

\[
|1\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad |2\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad |3\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}
\]

\[
|1^{'}\rangle \leftrightarrow \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad |2^{'}\rangle \leftrightarrow \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad |3^{'}\rangle \leftrightarrow \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}
\]

It is fairly obvious that the first basis is an orthogonal basis. By direct calculation, you can quickly see that the second basis is *not* orthogonal.

As a digression, we note that the inner product can also be written as

\[
\langle A | B \rangle = \sum_{jk} A^*_{jk} B_{jk} = \sum_{jk} A^T_{kj} B_{jk} = \text{Tr}(A^\dagger B)
\]

where $M^\dagger_{jk} = M^*_{kj}$ and $\text{Tr}(M) = \sum_j M_{jj}$ for any matrix $M$.

Here, we begin to see where matrix multiplication can become useful in this vector space. But note that it only becomes useful as a way to calculate the inner product.
Dual Spaces and Dirac Notation

We have seen examples of representing vectors $|v\rangle$ as column matrices for $\mathbb{R}^N$ and $\mathbb{C}^N$. This kind of column matrix representation is valid for any linear vector space because the space of column matrices, with standard column-matrix addition and scalar-column-matrix multiplication and scalar addition and multiplication, is itself a linear vector space. Essentially, column matrices are just a bookkeeping tool for keeping track of the coefficients of the basis elements.
When we begin to consider inner product spaces, we are naturally led to the question of how the inner product works in this column-matrix representation. We immediately see

$$\langle v \mid w \rangle = \sum_{j,k=1}^{N} v_j^* w_k \langle j \mid k \rangle$$

$$= \begin{bmatrix} v_1^* & \cdots & v_N^* \end{bmatrix} \begin{bmatrix} \langle 1 \mid 1 \rangle & \cdots & \langle 1 \mid N \rangle \\ \vdots & \ddots & \vdots \\ \langle N \mid 1 \rangle & \cdots & \langle N \mid N \rangle \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$$

That is, there is an obvious matrix representation of the inner product operation. When the basis is orthonormal, the above simplifies to

$$\langle v \mid w \rangle = \sum_{j=1}^{N} v_j^* w_j = \begin{bmatrix} v_1^* & \cdots & v_N^* \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$$
Purely for calculational and notational convenience, the above equation for the orthonormal basis case leads us to define, for any vector space $\mathbb{V}$, a partner space, called the dual space $\mathbb{V}^*$, via a row-matrix representation. That is, for a vector $|v\rangle$ in $\mathbb{V}$ with its standard column-matrix representation

$$
|v\rangle \leftrightarrow \begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix}
$$

we define a dual vector $\langle v |$ in the dual space $\mathbb{V}^*$ by its row-matrix representation

$$
\langle v | \leftrightarrow \begin{bmatrix}
v_1^* \\
v_2^* \\
\vdots \\
v_n^*
\end{bmatrix}
$$

A key point is that the dual space $\mathbb{V}^*$ is not identical to the vector space $\mathbb{V}$ and is not a vector space because the rules for scalar-vector multiplication are different: since there is a complex conjugation in the definition of the row-matrix representation, $\langle \alpha \cdot v | = \langle v | \alpha^*$ holds. (The placement of the $\alpha^*$ makes no difference to the meaning of the expression; we place the $\alpha^*$ after $\langle v |$ for reasons to be discussed soon.)
(Though $V^*$ is not a vector space, we might consider simply defining a dual vector space to be a set that satisfies all the vector space rules except for the complex conjugation during scalar-vector multiplication. It would be a distinct, but similar, mathematical object.)

Those with strong mathematical backgrounds may not recognize the above definition. The standard definition of the dual space $V^*$ is the set of all linear functions from $V$ to its scalar field; i.e., all functions on $V$ that, given an element $|v\rangle$ of $V$, return a member $\alpha$ of the scalar field associated with $V$. These functions are also called linear functionals, linear forms, one-forms, or covectors. We shall see below why this definition is equivalent to ours for the cases we will consider. If you are not already aware of this more standard definition of dual space, you may safely ignore this point!
With the definition of the dual space, and assuming we have the expansions
\(|v\rangle = \sum_{j=1}^{N} v_j |j\rangle \) and \(|w\rangle = \sum_{j=1}^{N} w_j |j\rangle \) in terms of an orthonormal basis for \(\mathbb{V}\), we may now see that the inner product \(\langle v | w \rangle \) can be written as the matrix product of the row-matrix representation of the dual vector \(\langle v |\) and the column-matrix representation of the vector \(|w\rangle\):

\[
\langle v | w \rangle = \sum_{j} v_j^* w_j = \begin{bmatrix} v_1^* & v_2^* & \cdots & v_n^* \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \langle v || w \rangle \quad (3.13)
\]

Again, remember that the representations of \(\langle v |\) and \(|w\rangle\) in terms of matrices are how our initial definitions of them are made, and are convenient for calculational purposes, but the representations are just that, representations; they are not the same thing as \(\langle v |\) and \(|w\rangle\), the latter have a more abstract existence.
The above now explains our comment about the dual space being the space of linear functionals: there is a one-to-one correspondence between the \( \langle v \mid \) dual vectors and the linear functionals \( \langle v \mid \) that accept a vector \( |w\rangle \) and returns a number \( \alpha \) by taking the inner product. In fact, one can show that any linear functional mapping from \( V \) to the field can be decomposed in terms of inner-product operations \( \langle v \mid \) . Mathematicians use the linear functional definition because it is more generic and connects to other concepts; for example, one-forms more easily generalize to vector spaces with \textit{curvature}, which we will most definitely not discuss in this course, and are connected to the differential curl operator. The most likely place you will encounter such objects are in a course in general relativity. I’ll bet, though that, like me, most of you can live without appreciating this subtlety the first time through...

It is standard practice to denote the vector \( |v\rangle \) belonging to \( V \) as a \textit{ket} and the dual vector \( \langle v | \) belonging to \( V^* \) as a \textit{bra}. These definitions are termed \textit{Dirac notation}. Depending on the circumstances, we will use the dual space, the Dirac notation, or both naming schemes.
We note that the basis vectors and their corresponding dual vectors satisfy

\[
|j\rangle \leftrightarrow \begin{bmatrix}
0 \\
\vdots \\
0 \\
1 \\
\vdots \\
0
\end{bmatrix} \quad \quad \langle j| \leftrightarrow \begin{bmatrix}
0 & \cdots & 0 & 1 & 0 & \cdots & 0
\end{bmatrix}
\]

where each matrix is nonzero only in its $j$th element. The above lets us write

\[
|v\rangle = \sum_j v_j |j\rangle = \sum_j \langle j|v\rangle |j\rangle \quad \langle v| = \sum_j \langle j|v^*_j = \sum_j \langle j|\langle v|j\rangle
\]

where $v_j = \langle j|v\rangle$ and $v^*_j = \langle v|j\rangle$ simply follow from the expansion of $|v\rangle$ in terms of $\{|j\rangle\}$. 

Example 3.14: Spin-1/2 particle at the origin

Let’s list the matrix representations of some vectors and their dual vectors (kets and bras) for the sake of being explicit:

\[
\begin{align*}
|\uparrow_x\rangle & \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \langle \uparrow_x | & \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \\
|\downarrow_x\rangle & \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \langle \downarrow_x | & \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix} \\
|\uparrow_y\rangle & \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} & \langle \uparrow_y | & \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \end{bmatrix} \\
|\downarrow_y\rangle & \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} & \langle \downarrow_y | & \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \end{bmatrix} \\
|\uparrow_z\rangle & \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \langle \uparrow_z | & \leftrightarrow \begin{bmatrix} 1 & 0 \end{bmatrix} \\
|\downarrow_z\rangle & \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \langle \downarrow_z | & \leftrightarrow \begin{bmatrix} 0 & 1 \end{bmatrix}
\end{align*}
\]
We can check the same inner products we did before, this time evaluating the inner products using the matrix representation (Equation 3.13) rather than representation-free sum over products of coefficients (Equation 3.5):

\[
\langle \uparrow y | \uparrow y \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{2} (1 + 1) = 1 = \langle \uparrow y | \uparrow y \rangle
\]

\[
\langle \downarrow y | \uparrow y \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{2} (1 - 1) = 0 = \langle \downarrow y | \uparrow y \rangle
\]

\[
\langle \downarrow y | \uparrow x \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} (1 + i) = \langle \downarrow y | \uparrow x \rangle
\]
We can now derive the linear expansions we wrote in Example 3.7: we use Equation 3.15 along with evaluation of the inner products using the matrix representation; e.g.,

\[ |\downarrow_z\rangle = \sum_j \langle j | \downarrow_z \rangle |j\rangle = \langle \uparrow_y | \downarrow_z \rangle |\uparrow_y\rangle + \langle \downarrow_y | \downarrow_z \rangle |\downarrow_y\rangle \]

\[ = \left( \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) |\uparrow_y\rangle + \left( \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) |\downarrow_y\rangle \]

\[ = \frac{-i}{\sqrt{2}} [ |\uparrow_y\rangle - |\downarrow_y\rangle] \]

This is a good example of a situation in which one has to avoid getting confused about what should be written out in matrix representation and what should not. The inner products \( \langle \uparrow_y | \downarrow_z \rangle \) and \( \langle \downarrow_y | \downarrow_z \rangle \) are written as matrix products. We could replace \( |\uparrow_y\rangle \) and \( |\downarrow_y\rangle \) by their matrix representations also. But keep in mind two things: 1) if you replace the vectors on the right side of the equation by column matrix representation, you must do the same on the left side, too: vectors and their representations are not the same thing; 2) the matrices making up the inner product do not act on the column matrix representation of the vectors by matrix multiplication, as indicated by the parentheses in the expression; the scalar result of the inner product multiplies the column matrices for the vectors.
Example 3.15: The set of real, antisymmetric $N \times N$ matrices, as in Example 3.5.

This is a particularly interesting example because you have to confront the many representations a particular inner product space can have. Let's consider the orthonormal basis we wrote down for the $N = 3$ case in Example 3.9:

$$|1\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad |2\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad |3\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Now, let's construct two new elements of the space via linear combinations; you will recognize these as the $|1'\rangle$ and $|2'\rangle$ normalized but non-orthogonal elements we previously constructed:

$$|1'\rangle = \frac{1}{\sqrt{2}} (|2\rangle + |3\rangle) \leftrightarrow \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$|2'\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle) \leftrightarrow \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$
Let’s consider three ways of taking the inner product $\langle 1′ | 2′ \rangle$. The first is the obvious way, using the explicit definition we had for the inner product for this space in terms of the defining representations:

$$
\langle 1′ | 2′ \rangle = \sum_{j,k=1}^{3} (1′)_{jk}(2′)_{jk}
$$

$$
= \frac{1}{4} \left[ 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 + (-1) \cdot (-1) + 0 \cdot 0 + 1 \cdot 0 + 0 \cdot (-1) + (-1) \cdot 0 + 0 \cdot 0 \right]
$$

$$
= \frac{1}{2}
$$

This above makes use of the representation we used to define the space, but makes no use of the generic column- and row-matrix representations we developed for an arbitrary inner product space with an orthonormal basis.
Now, let’s use the representation-free sum over coefficients, Equation 3.5, which makes use of the orthonormality of the basis but not the matrix representation of the vector and dual vector space:

\[ \langle 1' | 2' \rangle = \sum_{j=1}^{3} (1')_j (2')_j = \frac{1}{\sqrt{2}} (0 \cdot 1 + 1 \cdot 1 + 1 \cdot 0) = \frac{1}{2} \]

Finally, let’s write it out in terms of a matrix product of the matrix representations of the vector and dual vector spaces:

\[ \langle 1' | 2' \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \]

Thus, we see that there are two different matrix representations of this space: the one we used to define the space, and the one that appears when an orthonormal basis is used for the space. These are different in that they don’t look the same; but they are the same in that all the operations we have defined in one representation can be carried over to the other and vice versa in a consistent fashion. Clearly, though, it can be confusing to represent the same abstract object \( |v\rangle \) in two different ways — as an \( N \times N \) real, antisymmetric matrix and as a \( N(N-1)/2 \)-element column matrix — but this a key concept you must become accustomed to.
Representations as a Tool for Simplification and Unification

Hopefully, the previous two examples have illustrated the value of the matrix representation of inner product spaces once an orthonormal basis has been established — once you have established an orthonormal basis and expanded all the elements of the space in terms of that basis, you know the column-matrix representation of any element of the inner product space (and the row-matrix representation for its dual vector) and it is, frequently, arithmetically easier to take inner products using the matrix representation than to use the defining rule for the inner product in the space. Essentially, by recognizing the simpler underlying structure present once an orthonormal basis has been defined, we simplify the operations we must do on the space.

Another point is that the use of matrix representations allows us to unify different spaces, to realize that they are the same in spite of the apparent differences in the way they are defined. Mathematically, this is termed an isomorphism; the spaces are said to be isomorphic. In particular, any inner product space of dimension $N$ with a real (complex) field looks like $\mathbb{R}^N$ ($\mathbb{C}^N$) as far as any of the properties of the inner product space go. Of course, once one introduces additional operations on some spaces, this isomorphism may not be carried through to those operations. But the idea of isomorphism and the isomorphism of inner product spaces corresponding to different physical objects will be a theme we will return to repeatedly in this course.
Adjoints

We define the process of converting from vector to dual vector (ket to bra) and vice versa as taking the adjoint. \( \langle v \mid \) is the adjoint of \( |v\rangle \) and vice versa. In terms of the orthonormal basis matrix representation, there is a simple algorithm for this: complex conjugate and transpose.

From the above definition, the properties of complex numbers, and the definition of inner product, we can derive rules for taking the adjoint of any combination of bras, kets, and scalar coefficients:

- **scalar coefficients**: When one encounters a bra or ket with a scalar coefficient, the scalar coefficient must be complex conjugated (in addition to taking the adjoint of the bra or ket); i.e., the adjoint of \( \alpha |v\rangle \) is \( \langle v |\alpha^* \rangle \) and vice versa. Again, the placement of \( \alpha^* \) on the right is purely notational.
inner products: To determine the adjoint of an inner product $\langle v | w \rangle$, we use the fact that the inner product is just a complex number and so taking the adjoint of the inner product just corresponds to complex conjugation. But we know from the definition of inner product that complex conjugation of an inner product corresponds to exchanging the positions of the two vectors, so we see that the adjoint of $\langle v | w \rangle$ is $\langle w | v \rangle$. Thus, when we encounter inner products of bras and kets, we take the adjoint by simply reversing the order and converting bras to kets and vice versa, consistent with our rule for bras and kets alone with the addition of order-reversal. The need for order reversal is why we place scalar coefficients of bras to their right; the notation is now consistent.

sums: the adjoint of a sum is just the sum of the adjoints because complex conjugation and matrix transposition both behave this way.

products: Suppose one has an arbitrary product of inner products, scalar coefficients, and a bra or ket. (There can be nothing more complicated because the result would not be a bra or ket and hence could not be in the vector space or the dual vector space.) Our rules above simply imply that one should reverse the order of all the elements and turn all bras into kets and vice versa, even the ones in inner products. That is, for the ket

$$|u\rangle = \alpha_1 \cdots \alpha_k |w_1\rangle \cdots |w_m\rangle |v\rangle$$

(3.16)
where the \( \{ \alpha_j \} \), \( \{ | w_j \rangle \} \), and \( \{ | v_j \rangle \} \) are arbitrary (the index matchups mean nothing), we have that the adjoint is

\[
\langle u | = \langle v | \langle v_1 | w_1 \rangle \cdots \langle v_m | w_m \rangle \alpha_1^* \cdots \alpha_k^*
\]

(3.17)

**vector and dual vector expansions:** We may write our vector and dual vector expansions as

\[
| v \rangle = \sum_j \langle j | v \rangle | j \rangle = \sum_j | j \rangle \langle j | v \rangle \quad \quad \langle v | = \sum_j \langle v | j \rangle \langle j | \quad \quad (3.18)
\]

where we have simply exchanged the order of the inner product and the bra or ket; this is fine because the inner product is just a scalar. We see that the above expansions are fully consistent with our rules for taking adjoints of sums and products.