Linear Operators

Prologue

Now that we have defined inner product spaces, we have largely completed the work of defining the space that the states of a physical system live in. This is not enough, as physical states are not static. To make measurements and to obtain the dynamics of the system, we need *operators* that transform states into other states. According to postulates 2, 3, and 4, operators tell us how to carry classical mechanics over to quantum mechanics, how measurements work and how they affect states, and how to time-evolve states.
An operator $\Omega$ transforms a vector into a (possibly the same) vector, $\Omega |v\rangle = |w\rangle$ and transforms a dual vector into a (also possibly the same) dual vector, $\langle v |\Omega = \langle u |$.

Note that the action of the operator on $\langle v |$ is not necessarily the bra corresponding to the operation of the operator on $|v\rangle$; i.e., $\langle \Omega v | \neq \langle v |\Omega$ in general (though it will be true in some cases). This is why we stated above $\langle v |\Omega = \langle u |$ instead of $\langle v |\Omega = \langle w |$. 

Linear Operators (cont.)

However, it is also true that, once the operation of $\Omega$ on vectors has been set, there is no freedom in the action of $\Omega$ on dual vectors. One sees this as follows: suppose

$$\Omega |v_1\rangle = |w_1\rangle \quad \Omega |v_2\rangle = |w_2\rangle \quad \langle v_1 | \Omega = \langle u_1 | \quad \langle v_2 | \Omega = \langle u_2 |$$

Then we have the 4 independent relations

$$\langle v_1 | w_1 \rangle = \langle v_1 | \Omega | v_1 \rangle = \langle u_1 | v_1 \rangle \quad \langle v_1 | w_2 \rangle = \langle v_1 | \Omega | v_2 \rangle = \langle u_1 | v_2 \rangle$$

$$\langle v_2 | w_1 \rangle = \langle v_2 | \Omega | v_1 \rangle = \langle u_2 | v_1 \rangle \quad \langle v_2 | w_2 \rangle = \langle v_1 | \Omega | v_1 \rangle = \langle u_2 | v_2 \rangle$$

More generally, given $N$ linearly independent vectors in a $N$-dimensional vector space, there will be $N^2$ relations of the above type. Specifying the action of $\Omega$ on these $N$ vectors requires $2N^2$ numbers (the expansion coefficients of the $\{|u_j\rangle\}$ and $\{|w_j\rangle\}$). The $N^2$ expansion coefficients of the $\{|w_j\rangle\}$ were set when the action of $\Omega$ on the vector space was defined. The $N^2$ relations thus determine the $N^2$ expansion coefficients of the $\{|u_j\rangle\}$ from those of the $\{|w_j\rangle\}$. Thus, to determine the action of $\Omega$ on any $N$-element linearly independent set $\{\langle v_j |\}$, one need only look at $\Omega$’s action on the corresponding $\{|v_j\rangle\}$. Finally, if the action of $\Omega$ on the full set of vectors in the vector space is specified, then the action of $\Omega$ on the full set of dual vectors is specified by just picking linearly independent sets of the above type for each $|v\rangle$ and working out the relations.
We specialize to linear operators, those satisfying typical linearity relations:

\[ \Omega (\alpha \, |v\rangle + \beta \, |w\rangle) = \alpha \, \Omega |v\rangle + \beta \, \Omega |w\rangle \]

\[ (\langle v | \alpha + \langle w | \beta) \Omega = \alpha \langle v | \Omega + \beta \langle w | \Omega \]

(3.28)

Such operators are convenient, of course, because their action is completely specified by their action on the vector space’s basis vectors, which we shall come to momentarily. We specialize to linear operators because it is the simplest possible choice and it has been verified that quantum mechanics using only linear operators matches experimental predictions.

Our argument about the relation between the action of \( \Omega \) on vectors and dual vectors simplifies now: once the action of \( \Omega \) on an orthonormal basis \( \{|j\rangle\} \) has been specified, then our argument indicates that this specifies its action on the orthonormal basis \( \{\langle j |\} \). For a linear operator, specifying its action on an orthonormal basis then gives the action on the entire space by linearity, so the full action of \( \Omega \) on all vectors and dual vectors is specified.
Example 3.20: Identity Operator

That’s an easy one: for any $|v\rangle$, it returns $|v\rangle$:

$$I |v\rangle = |v\rangle$$

The only thing to point out here is that there is not just one identity operator; there is an identity operator for each vector space.
Example 3.21: Projection Operators

Given an inner product space $\mathbb{V}$ and a subspace $\mathbb{V}_P$, a projection operator is the operator that maps a vector $|v\rangle$ into its projection onto that subspace. An equivalent definition is: a projection operator is any operator of the form

$$P = \sum_j |v_j\rangle\langle v_j|$$  \hspace{1cm} (3.29)

where the $\{|v_j\rangle\}$ are members of an orthonormal basis (not necessarily all the members!). That is, each term calculates the inner product of the vector $|v\rangle$ it acts on with the unit vector $|v_j\rangle$, then multiplies the result by $|v_j\rangle$ to recover a vector instead of a number. One can see that the two definitions are equivalent by recognizing that the subspace $\mathbb{V}_P$ is just the space spanned by the set $\{|v_j\rangle\}$; alternatively, given the subspace $\mathbb{V}_P$, one should pick the $\{|v_j\rangle\}$ to be any orthonormal basis for $\mathbb{V}_P$. 
It is important for the vectors to be an orthonormal set in order to really pick out the piece of $|v\rangle$ in the subspace. To give a trivial counterexample, consider the vector $|v\rangle = v|j\rangle$, where $|j\rangle$ is an orthonormal basis element, and the projection operator $P = |v\rangle\langle v|$. Clearly, the output vector is always a vector in the subspace $\mathbb{V}_j$ spanned by $|j\rangle$ because $|v\rangle$ is in that subspace. But, let's act on $|v\rangle$ with $P$:

$$P|v\rangle = |v\rangle\langle v|v\rangle = |v|^2|v\rangle = |v|^2v|j\rangle$$

Since the original projection operator was not composed of normalized vectors, the normalization of the result is funny: it is not the projection of $|v\rangle$ onto the $\mathbb{V}_j$ subspace, but rather $|v|^2$ times that projection.
The above is just a normalization problem. A larger problem arises when one considers a projection operator composed of two non-orthonormal vectors. For example, in $\mathbb{R}^2$, consider the projection operator

$$P = |\hat{x}\rangle\langle \hat{x}| + \frac{\hat{x} + \hat{y}}{\sqrt{2}} \langle \frac{\hat{x} + \hat{y}}{\sqrt{2}} |$$

where $|\hat{x}\rangle$ and $|\hat{y}\rangle$ are the $x$ and $y$ unit vectors. The vectors used to construct $P$ are normalized but not orthogonal. The subspace spanned by the vectors making up the operator is the entire space, $\mathbb{R}^2$, because the two vectors are linearly independent and the space is already known to be 2-dimensional. Let’s try acting on $|\hat{y}\rangle$ (using antilinearity of the bra):

$$P|\hat{y}\rangle = |\hat{x}\rangle\langle \hat{x}| \hat{y} \rangle + \frac{\hat{x} + \hat{y}}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} (\langle \hat{x} | \hat{y} \rangle + \langle \hat{y} | \hat{y} \rangle) = \frac{1}{\sqrt{2}} \frac{\hat{x} + \hat{y}}{\sqrt{2}} \right)$$

Since the subspace spanned by the vectors making up the operator is the space, this projection operator ought to have returned $|\hat{y}\rangle$; it did not.
We can explicitly see that, if the projection operator is composed of an orthonormal set, it does indeed recover the portion of $|v\rangle$ in the subspace spanned by that set. Let's consider an orthonormal basis $\{|j\rangle\}$ of a $n$-dimensional inner product space, and let's consider a projection operator onto the subspace $\mathbb{V}_1 \ldots m$ spanned by the first $m$ of the basis elements. (We can always reorder the basis elements so the ones that we want to use are the first $m$.) That projection operator is

$$P = \sum_{j=1}^{m} |j\rangle \langle j|$$

Acting on an arbitrary $|v\rangle$ in the space (with expansion $\sum_{j=1}^{n} v_j |j\rangle$) with $P$ thus gives

$$P|v\rangle = \left( \sum_{j=1}^{m} |j\rangle \langle j| \right) \left( \sum_{k=1}^{n} v_k |k\rangle \right) = \sum_{j=1}^{m} \sum_{k=1}^{n} v_k |j\rangle \langle j| k\rangle = \sum_{j=1}^{m} \sum_{k=1}^{n} v_k |j\rangle \delta_{jk}$$

$$= \sum_{j=1}^{m} v_j |j\rangle$$

which is, by our original expansion of $|v\rangle$, the piece of $|j\rangle$ in the subspace spanned by the first $m$ of the $\{|j\rangle\}$.
Linear Operators (cont.)

It is frequently useful to rewrite the identity operator in terms of projection operators:

\[ I = \sum_{j=1}^{n} |j\rangle \langle j| \]  

(3.30)

That this sum is indeed the identity operator can be seen by using the same proof we just made above but taking \( m = n \). Then the result is

\[
\left[ \sum_{j=1}^{n} |j\rangle \langle j| \right] |v\rangle = \left[ \sum_{j=1}^{n} |j\rangle \langle j| \right] \sum_{k=1}^{n} v_k |k\rangle = \sum_{j=1}^{n} v_j |j\rangle = |v\rangle
\]

It follows from the above that a projection operator \( P \) that projects onto the subspace \( \mathbb{V}_P \) is the identity operator \( I_P \) for \( \mathbb{V}_P \). When \( \mathbb{V}_P = \mathbb{V} \), one recovers \( I \) for the full space \( \mathbb{V} \).

A final note: frequently, we will use the subscript \( j \) to denote projection on the subspace spanned by the single orthonormal basis element \( |j\rangle \):

\[ P_j = |j\rangle \langle j| \]

Of course, \( P_j \) is not specified until you specify a basis, so the meaning of \( P_j \) will always depend on context. But this is standard notation.
Example 3.22: Derivative Operator on Function Spaces on Discrete Points

Returning to the Example 3.4, let’s create something that looks like taking the derivative. Let’s use the orthonormal basis \{ |j\rangle \} consisting of the functions that are 1 at \( x_j \) and 0 elsewhere (i.e., just like the standard basis of \( \mathbb{C}^N \)). Then define

\[
D_R |j\rangle = -\frac{|j\rangle - |j-1\rangle}{\Delta} \quad \text{for} \quad j \neq 1 \quad D_R |1\rangle = -\frac{|1\rangle - |N\rangle}{\Delta} \quad x_j = j \frac{L}{N+1} = j \Delta
\]

Then we have

\[
D_R |f\rangle = D_R \sum_{j=1}^{N} f(x_j)|j\rangle = -\frac{1}{\Delta} \left[ f(x_1) (|1\rangle - |N\rangle) + \sum_{j=2}^{N} f(x_j) (|j\rangle - |j-1\rangle) \right]
\]

\[
= \left[ \sum_{j=1}^{N-1} \frac{f(x_{j+1}) - f(x_j)}{\Delta} |j\rangle \right] + \frac{f(x_1) - f(x_N)}{\Delta} |N\rangle
\]

The output function looks like the right-going discrete derivative of \( f(x_j) \)! We looped around the end at the last point; that would not be necessary in the limit \( \Delta \to 0 \). You are well aware from calculus that taking the derivative of continuous functions is a linear operation, the same holds here.
Example 3.23: Spin-1/2 Operators

Let’s give our first example of an operator that returns an observable, the spin projection of spin-1/2 particle states. We simply take as definitions

\[ S_z |\uparrow_z\rangle = \frac{\hbar}{2} |\uparrow_z\rangle \quad S_z |\downarrow_z\rangle = -\frac{\hbar}{2} |\downarrow_z\rangle \]  \hspace{1cm} (3.31)
\[ S_x |\uparrow_x\rangle = \frac{\hbar}{2} |\uparrow_x\rangle \quad S_x |\downarrow_x\rangle = -\frac{\hbar}{2} |\downarrow_x\rangle \]  \hspace{1cm} (3.32)
\[ S_y |\uparrow_y\rangle = \frac{\hbar}{2} |\uparrow_y\rangle \quad S_y |\downarrow_y\rangle = -\frac{\hbar}{2} |\downarrow_y\rangle \]  \hspace{1cm} (3.33)

We have really put the cart before the horse here because we never explained why the states that we defined as \(|\uparrow_z\rangle, |\downarrow_z\rangle, |\uparrow_x\rangle\), etc. corresponded to physical states with spin projection along \(+z, -z, +x\), etc. But neglecting that motivational problem, which we will deal with later, it is clear that the above is a perfectly valid definition of a set of operators, and they now have some physical meaning: these operators tell us the spin projection along particular axes of particular states.
We have only specified the spin operators in terms of the states they leave unchanged (up to normalization). These states are a complete basis for the space in each case, so this is sufficient. But let us look at how they change other states. For example, using the above and some results derived in Example 3.14:

\[ S_y \downarrow_z = S_y \frac{-i}{\sqrt{2}} (|\uparrow_y\rangle - |\downarrow_y\rangle) = -\frac{i\hbar}{2\sqrt{2}} (|\uparrow_y\rangle + |\downarrow_y\rangle) = -\frac{i\hbar}{2} |\uparrow_z\rangle \]

That is, \( S_y \) converts \( |\downarrow_z\rangle \) to \( |\uparrow_z\rangle \), modulo a normalization factor. We will make use of the “raising” behavior later. For now, it simply serves to show that, having defined the action of \( S_y \) on the \( |\uparrow_y\rangle \) and \( |\downarrow_y\rangle \) basis states, we can now calculate the action on any state, a point that we will state more generally next. The same holds for \( S_x \) and \( S_z \).

**Example 3.24: Rotation Operators in \( \mathbb{R}^3 \).**

Another example, discussed by Shankar, is that of rotation operators in \( \mathbb{R}^3 \). Read this example.
Linear Operators (cont.)

Linear Operator Action on Basis Vectors, Matrix Elements, Matrix Representation of Linear Operators

The main advantage to staying with linear operators is that their action on any vector is defined purely by their action on a set of basis vectors. Given a set of basis vectors \{ |j\rangle \} and a vector \( |j\rangle = \sum_j v_j |j\rangle \) expanded in terms of them, we may write

\[
\Omega |v\rangle = \Omega \sum_j v_j |j\rangle = \sum_j v_j \Omega |j\rangle \quad (3.34)
\]

It is useful to break \( \Omega |j\rangle \) into components by rewriting the expansion using the identity operator written out using projection operators, \( I = \sum_k |k\rangle \langle k| \):

\[
\Omega |v\rangle = \sum_j v_j \Omega |j\rangle = \sum_j v_j \sum_k |k\rangle \langle k| \Omega |j\rangle
\]

We define the projection of \( \Omega |j\rangle \) onto \( |k\rangle \) as \( \Omega_{kj} \), \( \Omega_{kj} = \langle k | \Omega |j\rangle \). These are just numbers. The expression can then be rewritten

\[
\Omega |v\rangle = \sum_{jk} \Omega_{kj} v_j |k\rangle = \sum_{jk} |k\rangle \Omega_{kj} v_j
\]

thereby giving the components of the result along the various \{ |k\rangle \}. 
The above expression looks like matrix multiplication of a $n \times n$ matrix against a single-column $n$-row matrix:

\[
[\Omega |v\rangle]_k = \langle k |\Omega |v\rangle = \sum_j \Omega_{kj} v_j
\]  (3.35)

This makes sense: we were able to represent our vectors via single-column matrices (kets) and our dual vectors as single-row matrices (bras); it is consistent for operators to be represented as $n \times n$ matrices (where $n$ is the dimensionality of the vector space) and the $kj$ element ($k$th row, $j$th column) is just the projection of the action of $\Omega$ on $|j\rangle$ onto $|k\rangle$. We have thus found the matrix representation of the operator $\Omega$ in the column-matrix representation of the vector space with orthonormal basis \{\{|j\rangle\}\}. 

We may of course derive similar relations for the operation of $\Omega$ on a bra:

$$\langle v | \Omega = \sum_j \langle j | \Omega v_j^* = \sum_{jk} v_j^* \langle j | \Omega | k \rangle \langle k | = \sum_{jk} v_j^* \Omega_{jk} \langle k |$$

or
$$[\langle v | \Omega]_k = \sum_j v_j^* \Omega_{jk} \tag{3.36}$$

which again looks like matrix multiplication, this time of a $n \times n$ matrix on a single-row, $n$-column matrix on its left. $\Omega_{jk}$ is the projection of the action of $\Omega$ on $\langle j |$ onto $\langle k |$ (note the transposition of the indices relative to the ket case). The matrix representation of $\Omega$ is thus also consistent with the row-matrix representation of the dual vector space with orthonormal basis $\{\langle j |\}$.

We note that the above relation corroborates the statement we made at the start of our discussion of operators that specifying the action of $\Omega$ on a linear basis for $\mathbb{V}$ also fully determines its action on a linear basis for $\mathbb{V}^*$. Here, the $\Omega_{kj}$ are the $N^2$ numbers that give the action of $\Omega$ on any ket $|v\rangle$, as indicated in Equation 3.35. But these same $N^2$ numbers appear in Equation 3.36, which expresses the action of $\Omega$ on any bra $\langle v |$. 

Section 3.5 Mathematical Preliminaries: Linear Operators
Let us summarize our point about matrix representations of operators: given a linear operator $\Omega$ on a $n$-dimensional inner product space $\mathbb{V}$ with an orthonormal basis $\{|j\rangle\}$ (and corresponding dual space with orthonormal basis $\langle j|\rangle$), we may write a $n \times n$ matrix representation of the operator $\Omega$ with elements $\Omega_{kj}$ given by

$$
\Omega_{kj} = \langle k | \Omega | j \rangle
$$

(3.37)

and the action of this matrix on the column-matrix representation of $\mathbb{V}$ and the row-matrix representation of $\mathbb{V}^*$ is consistent with the operation of $\Omega$ on the elements of $\mathbb{V}$ and $\mathbb{V}^*$. Matrix representations of operators will be the tool we use to do much of quantum mechanics.
Example 3.25: Projection Operators Revisited

Projection operators have a very simply matrix representation:

\[(P_j)_{kl} = \langle k | P_j | m \rangle = \langle k | j \rangle \langle j | m \rangle = \delta_{kj} \delta_{jm} \quad (3.38)\]

That is, \(P_j\) is an empty matrix except for a 1 in the \(jj\) element. Conveniently, this extends the consistency of the matrix representation scheme for bras and kets if we define an outer product between vectors,

\[
|v\rangle \langle w | = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} w_1^* & w_2^* & \cdots & w_n^* \end{bmatrix} = \begin{bmatrix} v_1 w_1^* & v_1 w_2^* & \cdots & v_1 w_n^* \\ v_2 w_1^* & v_2 w_2^* & \cdots & v_2 w_n^* \\ \vdots & \vdots & \ddots & \vdots \\ v_n w_1^* & v_n w_2^* & \cdots & v_n w_n^* \end{bmatrix} \quad (3.39)
\]

or

\[
|v\rangle \langle w |_{km} = v_k w_m^* \quad (3.40)
\]

because, for a projection operator \(P_j = |j\rangle \langle j|\), we have \(v_k = \delta_{kj}\) and \(w_m = \delta_{jm}\) as shown earlier.
For $\mathbb{R}^3$ with the standard $x$, $y$, $z$ basis for the column-matrix representation, the projection operators that project onto the $x$, $y$, and $z$ axes are

$$P_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The projection operators into various planes are

$$P_{xy} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P_{yz} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_{xz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
We note that one can write a projection operator in a matrix representation corresponding to a basis that does not match the natural basis of the projection operator. For example, in the above basis, the operator that projects onto the subspace defined by the vector \((\hat{x} + \hat{y})/\sqrt{2}\) is

\[
P_{\hat{x}+\hat{y}} = \left( \frac{\langle \hat{x} \rangle + \langle \hat{y} \rangle}{\sqrt{2}} \right) \left( \frac{\langle \hat{x} \rangle + \langle \hat{y} \rangle}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ \end{bmatrix}
\]

\[
= \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

The key is to use the outer product form, writing the column- and row-matrix representations of the bras and kets making up the outer product.
Example 3.26: Derivative Operator on Function Spaces on Discrete Points, as in Example 3.22

We defined the action of the derivative operator on the space of functions on $N$ discrete points as follows:

$$D_R|j\rangle = \begin{cases} -(|1\rangle - |N\rangle)/\Delta & j = 1 \\ -(|j\rangle - |j-1\rangle)/\Delta & j \neq 1 \end{cases}$$

$$x_j = j \frac{L}{N+1} = j \Delta$$

It’s easy to calculate the matrix elements:

$$(D_R)_{kj} = \langle k | D_R | j \rangle = \begin{cases} (\delta_{k,N} - \delta_{k,1})/\Delta & j = 1 \\ (\delta_{k,j-1} - \delta_{k,j})/\Delta & j \neq 1 \end{cases}$$

Let’s check that we get the expected result from this matrix representation for $N = 4$:

$$D_R|f\rangle \leftrightarrow \frac{1}{\Delta} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{bmatrix} = \begin{bmatrix} [f(x_2) - f(x_1)]/\Delta \\ [f(x_3) - f(x_2)]/\Delta \\ [f(x_4) - f(x_3)]/\Delta \\ [f(x_1) - f(x_4)]/\Delta \end{bmatrix}$$

You have to be careful about what is meant by $\delta_{k,j-1}$. For example, for row 1 of the matrix, $k = 1$, we have $\delta_{k,j-1} = 1$ when $j - 1 = k = 1$, so $j = 2$ has the nonzero element.
Example 3.27: Spin-1/2 Operators

Let's write out the matrix representations of the $S_x$, $S_y$, and $S_z$ operators in the orthonormal basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$. We state without derivation that they are

$$S_z \leftrightarrow \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad S_x \leftrightarrow \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad S_y \leftrightarrow \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (3.41)$$

Checking $S_z$ is easy because we defined it by its action on $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$, which are the elements of the orthonormal basis we are using for this matrix representation:

$$S_z |\uparrow_z\rangle \leftrightarrow \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \leftrightarrow \frac{\hbar}{2} |\uparrow_z\rangle$$

$$S_z |\downarrow_z\rangle \leftrightarrow \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \leftrightarrow \frac{\hbar}{2} |\downarrow_z\rangle$$

which reproduce Equation 3.31.
We have defined $S_{x}$ and $S_{y}$ in terms of their action on $|\uparrow_{x}\rangle$, $|\downarrow_{x}\rangle$ and $|\uparrow_{y}\rangle$, $|\downarrow_{y}\rangle$, respectively, so one must apply the matrix representations of the operators in this basis to the matrix representations of those vectors in this basis. The latter were given in Example 3.14. Let’s try one example here:

$$S_{y}|\downarrow_{y}\rangle \leftrightarrow \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} = -\frac{\hbar}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} \leftrightarrow -\frac{\hbar}{2}|\downarrow_{y}\rangle$$

which matches Equation 3.33.