Lecture 7:
Linear Operators Continued
Eigenvalue-Eigenvector Problem
Date Revised: 2008/10/13
Date Given: 2008/10/13
Hermitian and Anti-Hermitian Operators: the Operators of QM

Hermitian and anti-Hermitian operators are defined by the relations

\[
\text{Hermitian: } \Omega^\dagger = \Omega \quad \text{anti-Hermitian: } \Omega^\dagger = -\Omega \tag{3.48}
\]

Remember that, by definition of operator adjoints, the above are equivalent to

\[
\text{Hermitian: } \Omega |v\rangle = |w\rangle \iff \langle v | \Omega = \langle w | \tag{3.49}
\]

\[
\text{anti-Hermitian: } \Omega |v\rangle = |w\rangle \iff \langle v | \Omega = -\langle w | \tag{3.49}
\]

The matrix representation versions of the above definitions are

\[
\text{Hermitian: } \Omega^*_{jk} = \Omega_{kj} \quad \text{anti-Hermitian: } \Omega^*_{jk} = -\Omega_{kj} \tag{3.50}
\]

A sum of two Hermitian operators is easily seen to be Hermitian. The product of two Hermitian operators need not be Hermitian because the two operators have their order reversed when the adjoint is taken: \((\Omega\Lambda)^\dagger = \Lambda^\dagger \Omega^\dagger = \Lambda\Omega \neq \Omega\Lambda\) in general.

Hermitian and anti-Hermitian operators are obvious analogues of purely real and purely imaginary numbers. At a qualitative level, it becomes clear why the operator for a classical physical variable must be Hermitian — we want our physical observables to be real numbers! We will of course justify this rigorously later.
Example 3.28: Projection Operators and Hermiticity

Let us show that our standard projection operator definition is Hermitian. Given a set of orthonormal vectors \( \{|j\rangle\} \), \( j = 1, \ldots, m \) that span the subspace \( \mathbb{V}_P \) of an \( n \)-dimensional inner product space that we want to project onto, we have

\[
P = \sum_{j=1}^{m} |j\rangle\langle j| \quad \iff \quad P^\dagger = \left( \sum_{j=1}^{m} |j\rangle\langle j| \right)^\dagger = \sum_{j=1}^{m} (|j\rangle\langle j|)^\dagger
\]

What is \( (|j\rangle\langle j|)^\dagger \)? We have an expectation that \( (|j\rangle\langle j|)^\dagger = |j\rangle\langle j| \) based on our definition of adjoint for kets and bras, but we have not explicitly showed that should be true when the combination of kets and bras is an operator, not a ket or a bra. So, let’s go back to the definition of an operator adjoint, Equation 3.44. In this case, it requires

\[
\text{if} \quad (|j\rangle\langle j|)|v\rangle = |w\rangle \quad \text{then} \quad \langle w| = \langle v| (|j\rangle\langle j|)^\dagger
\]

Let’s just rewrite the first expression:

\[
|j\rangle\langle j| |v\rangle = |w\rangle
\]
Take the adjoint (which we can do since there are no operators involved now):

$$\langle v | j \rangle \langle j | = \langle w |$$

This looks like the second half of our adjoint definition statement. That statement becomes true if

$$(|j\rangle \langle j|)^\dagger = |j\rangle \langle j|$$

which is what we expected, but now we have proven it explicitly. So, then,

$$P^\dagger = \sum_{j=1}^{m} (|j\rangle \langle j|)^\dagger = \sum_{j=1}^{m} |j\rangle \langle j| = P$$

Projection operators are Hermitian.
An alternate definition of a projection operator $P$ is that $P$ be Hermitian and that it satisfy the rather unobvious condition

$$P^2 = P \quad (3.51)$$

Let us show now that this definition is equivalent to our definition Equation 3.29.

First, we show that Equation 3.29 implies the above. We have already demonstrated that it implies Hermiticity. Now let us show it implies Equation 3.51. Again, let $\{|j\rangle\}$, $j = 1, \ldots, m$ be a set of orthonormal vectors that span the subspace $\mathbb{V}_P$ of a $n$-dimensional inner product space $\mathbb{V}$ that we want to project onto. Then

$$P^2 = \sum_{j=1}^{m} |j\rangle\langle j| \sum_{k=1}^{m} |k\rangle\langle k| = \sum_{j,k=1}^{m} |j\rangle\langle j| |k\rangle\langle k| = \sum_{j,k=1}^{m} |j\rangle \delta_{jk} \langle k| = \sum_{j=1}^{m} |j\rangle\langle j| = P$$

as desired.
Let us show the converse, that the conditions $P^2 = P$ and that $P$ is Hermitian imply our original definition Equation 3.29. The condition $P^2 = P$ implies that, for any $|v\rangle$, 

$$P(P|v\rangle) = P|v\rangle$$

Let $\mathbb{V}_P$ be the set of vectors produced by acting with $P$ on all $|v\rangle$ belonging to $\mathbb{V}$. We can see that this set is a subspace as follows. Suppose $|v'\rangle$ and $|w'\rangle$ belong to $\mathbb{V}_P$. By definition of $\mathbb{V}_P$, there must be (possibly non-unique) vectors $|v\rangle$ and $|w\rangle$ such that $|v'\rangle = P|v\rangle$ and $|w'\rangle = P|w\rangle$. Then the linear combination $\alpha|v'\rangle + \beta|w'\rangle$ satisfies $\alpha|v'\rangle + \beta|w'\rangle = \alpha P|v\rangle + \beta P|w\rangle = P(\alpha|v\rangle + \beta|w\rangle)$, thereby implying that the linear combination belongs to the $\mathbb{V}_P$ also. So $\mathbb{V}_P$ is closed under all the necessary operations, so it is a subspace.

Now, for any element $|v'\rangle$ in the subspace $\mathbb{V}_P$, it holds that $P|v'\rangle = |v'\rangle$, as follows: For any such element $|v'\rangle$, there is at least one vector $|v\rangle$ such that $|v'\rangle = P|v\rangle$. Since we know $P(P|v\rangle) = P|v\rangle$, it therefore holds $P|v'\rangle = |v'\rangle$.

So, we have that $\mathbb{V}_P$ is a subspace and $P|v\rangle = |v\rangle$ for any $|v\rangle$ in $\mathbb{V}_P$. Let \{
\begin{align*}
|j\rangle
\end{align*}
\} be an orthonormal basis for this subspace, $j = 1, \ldots, m$ where $m$ is the dimension of the subspace. Then it holds that $P|j\rangle = |j\rangle$ for these $|j\rangle$. Therefore, $\langle k | P|j\rangle = \delta_{kj}$ for $j, k = 1, \ldots, m$. This gives us some of the matrix elements of $P$. 


Extend this orthonormal basis to be an orthonormal basis for the full space, \( j = 1, \ldots, n \) where \( n \) is the dimension of the full space. We know \( \langle j | k \rangle = \delta_{jk} \).

Therefore, for \( j = 1, \ldots, m \) and \( k = m + 1, \ldots, n \), it holds

\[
\langle k | (P|j\rangle) = |k\rangle (|j\rangle) = \langle k | j\rangle = \delta_{kj} = 0
\]

\[
\langle j | (P|k\rangle) = \langle j | P|k\rangle = \left(P^{\dagger}|j\rangle \right)^{\dagger} |k\rangle = (P|j\rangle)^{\dagger} |k\rangle = (|j\rangle)^{\dagger} |k\rangle = \langle j | k\rangle = \delta_{jk} = 0
\]

we used the definition of adjoint operators, bras, and kets and the assumed Hermiticity of \( P \). \( \delta_{jk} \) vanished in both cases because we had \( j = 1, \ldots, m \) and \( k = m + 1, \ldots, n \): \( j \) and \( k \) are never the same.

The last matrix elements we need are easy. We want to know what \( \langle k | P|k\rangle \) is for \( k = m + 1, \ldots, n \). Since \( P|k\rangle \) belongs to \( \mathbb{V}_P \) while \( |k\rangle \) is orthogonal to the orthonormal basis for \( \mathbb{V}_P \), this matrix element always vanishes.
To summarize,

\[ \langle j | P | k \rangle = \delta_{jk} \quad \text{for } j, k = 1, \ldots, m \]

\[ 0 \quad \text{otherwise} \]

We may then use the bilinear form to write out the explicit form for the projection operator:

\[ P = \sum_{j, k=1}^{n} |j\rangle \langle j | P | k \rangle \langle k | = \sum_{j, k=1}^{m} |j\rangle \delta_{jk} \langle k | = \sum_{j=1}^{m} |j\rangle \langle j | \]

which is Equation 3.29, our original definition of the projection operator for the subspace spanned by the orthonormal set \( \{|j\rangle\} \), \( j = 1, \ldots, m \).

We note that \( P^2 = P \) does not imply \( P \) is its own inverse. Projection operators are in general noninvertible. Let \( \mathbb{V}_P \) be the subspace of the inner product space \( \mathbb{V} \) onto which the projection operator \( P \) projects. Consider a vector \( |v\rangle \) in the subspace \( \mathbb{V}_P^\perp \) that is orthogonal to \( \mathbb{V}_P \), meaning that it is orthogonal to all the \( \{|j\rangle\} \) comprising \( P \). Then \( P|v\rangle = |0\rangle \). But \( P|0\rangle = |0\rangle \) also, so \( P \) is not one-to-one, and hence cannot be invertible. The only case for which this argument fails is for \( P = I \) because then the subspace \( \mathbb{V}_P^\perp \) has \( |0\rangle \) as its only element.
Example 3.29: Spin-1/2 Operators are Hermitian

You can see quite easily that $S_x$, $S_y$, and $S_z$ are Hermitian operators by simply taking the complex conjugate transpose of the matrix representations we have already given. For example,

$$S_y^\dagger \leftrightarrow \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$
Unitary Operators: Operators that do Transformations in QM

Unitary operators are defined by the relation

\[ U^\dagger = U^{-1} \quad (3.52) \]

By definition of operator adjoints, the above is equivalent to

\[ U|v\rangle = |w\rangle \iff \langle v|U^{-1} = \langle w| \quad (3.53) \]

We will obtain a definition in terms of matrix representations on the next page.

A product of unitary operators is unitary; one can see this by simply using the product rules for adjoints and inverses. The appropriately normalized sum of unitary operators need not be unitary: when one tests whether \((U_1 + U_2)(U_1 + U_2)^\dagger/4 = I\), one ends up with two cross terms that do not give \(I\) unless \(U_1 = U_2\).

Unitary operators are like complex numbers of unit modulus, \(e^{i\theta}\). Conjugating such a number gives its multiplicative inverse, just as taking the adjoint of a unitary operator gives its operator product inverse. In QM, unitary operators “transform” states — they time evolve them, spatially rotate them, etc. You can think of them as the analogue of the \(e^{i\omega t}\) and \(e^{ikx}\) factors in electromagnetic wave propagation, though of course their effect is more complicated than that. They are of complementary importance to Hermitian operators.
Inner Products and Unitary Operators

Unitary operators preserve inner products; i.e.,

\[ |v'\rangle = U |v\rangle \text{ and } |w'\rangle = U |w\rangle \text{ then } \langle w' | v' \rangle = \langle w | v \rangle \] (3.54)

The proof is trivial:

\[ \langle w' | v' \rangle = (U |w\rangle)^\dagger (U |v\rangle) = \langle w | U^\dagger U |v\rangle = \langle w | v \rangle \] (3.55)

We thus see that unitary operators are generalizations of rotation and other orthogonal operators from classical mechanics, which preserve the \( \mathbb{R}^3 \) dot product.

One property of orthogonal matrices that carries over to unitary operators is the orthonormality of their rows and columns in matrix representation, treating their columns as kets or rows as bras. Shankar gives two proofs; we give the matrix-arithmetic version to provide experience with such manipulations:

\[ \langle \text{column } j | \text{column } k \rangle = \sum_m U^*_m U_{mk} = \sum_m U^\dagger_{jm} U_{mk} = [U^\dagger U]_{jk} = \delta_{jk} \] (3.56)

The row version is similar. Orthonormality of the columns and rows implies the operator is unitary.
Unitary Transformations of Operators

As we noted, unitary operators transform states, such as for time evolution or spatial translation. One of the most basic questions we ask in QM is: how do the matrix elements of some operator change under such a transformation. The interest in the time evolution case is obvious; in other transformations, we are usually interested in how the transformation of operator matrix elements is related to symmetries of the problem.

Explicitly, we might ask: how is \( \langle w | \Omega | v \rangle \) related to \( \langle w' | \Omega | v' \rangle \) where \( |v'\rangle = U|v\rangle \) and \( |w'\rangle = U|w\rangle \)? Of course the specific answer depends on the problem. But it is generally true that the second expression may be written

\[
\langle w' | \Omega | v' \rangle = (U|w\rangle) \dagger \Omega (U|v\rangle) = \langle w | \left( U\dagger \Omega U \right) | v \rangle \tag{3.57}
\]

The states are now untransformed; instead, we consider the matrix elements of the transformed operator, \( \Omega' = U\dagger \Omega U \) between the untransformed states.
This concept has numerous applications. As we shall see next, we frequently would like to use a basis of eigenstates of some operator $H$ (states $|v\rangle$ for which $H|v\rangle = h|v\rangle$ where $h$ is a number). We can apply a unitary transformation to get from our initial basis to such a basis, and the above transformation lets us see how other operators are represented in the new basis.

Another application is time evolution. The standard picture is the Schrödinger picture, in which we apply a unitary time evolution operator to the states. In the alternate Heisenberg picture, we leave the states unchanged and apply the time-evolution transformation to operators.
Example 3.30: Particular Unitary Transformations of Spin-1/2 Vectors and Operators

Let’s consider a particular set of unitary operators our standard $\mathbb{C}^2$ matrix representation of this space:

$$U = e^{i\alpha} \begin{bmatrix} \cos \frac{\theta}{2} & -e^{-i\phi} \sin \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix}$$ (3.58)

The four column-row orthonormality conditions leave four degrees of freedom for the arbitrary unitary matrix in this representation. This can be represented as one free angle $\theta$ that is the argument of the cosines and sines combined with three free phase angles. We have taken one of the phase angles to vanish. Let’s try it out on our various states:

$$U(\alpha, \theta, \phi)|\uparrow_z\rangle \leftrightarrow e^{i\alpha} \begin{bmatrix} \cos \frac{\theta}{2} & -e^{-i\phi} \sin \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{i\alpha} \begin{bmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{bmatrix}$$

$$U(\alpha, \theta, \phi)|\downarrow_z\rangle \leftrightarrow e^{i\alpha} \begin{bmatrix} \cos \frac{\theta}{2} & -e^{-i\phi} \sin \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{i\alpha} \begin{bmatrix} e^{-i\phi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix}$$
In particular, we see that if we take $\theta = \pi$, then we obtain

$$U(\alpha, \theta = \pi, \phi)|\uparrow_z\rangle \leftrightarrow e^{i(\alpha+\phi)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \leftrightarrow e^{i(\alpha+\phi)}|\downarrow_z\rangle$$

$$U(\alpha, \theta = \pi, \phi)|\downarrow_z\rangle \leftrightarrow e^{i(\alpha-\phi)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \leftrightarrow e^{i(\alpha-\phi)}|\uparrow_z\rangle$$

This particular unitary operator has rotated the $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ basis elements into each other, up to unity modulus complex factors. With $\alpha = 0$ and $\phi = 0$, the exchange would be exact. This is equivalent to a spatial rotation of the physical space coordinate axes of $\pi$ radians about any vector in the $xy$ plane.
What about the action on the other possible bases?

\[
U(\alpha = 0, \theta = \pi, \phi = 0) |\uparrow_x\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \leftrightarrow -|\downarrow_x\rangle
\]

\[
U(\alpha = 0, \theta = \pi, \phi = 0) |\downarrow_x\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftrightarrow |\uparrow_x\rangle
\]

\[
U(\alpha = 0, \theta = \pi, \phi = 0) |\uparrow_y\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix} \leftrightarrow -i |\uparrow_y\rangle
\]

\[
U(\alpha = 0, \theta = \pi, \phi = 0) |\downarrow_y\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix} \leftrightarrow i |\downarrow_y\rangle
\]

We see that the physical space rotation is about the y axis, so that the transformation rotates \(|\uparrow_x\rangle\) and \(|\downarrow_x\rangle\) into each other (modulo signs) and keeps \(|\uparrow_y\rangle\) and \(|\downarrow_y\rangle\) unchanged (modulo unity modulus factors).
How about unitary transformation of operators? Again, using \( \alpha = 0, \theta = \pi, \) and \( \phi = 0, \) let’s apply the unitary transformation to \( S_x, S_y, \) and \( S_z: \)

\[
U^\dagger S_z U \leftrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \leftrightarrow -S_z
\]

\[
U^\dagger S_x U \leftrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \leftrightarrow -S_x
\]

\[
U^\dagger S_y U \leftrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \leftrightarrow S_y
\]

The sign flips on \( S_z \) and \( S_x \) make sense, as we saw that the corresponding basis states were rotated into each other, while the lack of change for \( S_y \) makes sense because we saw the unitary transformation left them unaffected except for prefactors of unity modulus.

Note that one *either* transforms the states *or* transforms the operators, not both — that’s why the exchange of \( |\uparrow_z\rangle \) and \( |\downarrow_z\rangle \) and the sign flip on \( S_z \) do not cancel one another because one does not both of them, one does only one, depending on whether you want to transform the states or the operators.
Motivation

An eigenvector of an operator is a vector that is left unchanged by the operator up to a scalar multiplier, which is called the eigenvalue. The vector $|\omega\rangle$ is an eigenvector of the operator $\Omega$ with eigenvalue $\omega$ if and only if

$$\Omega |\omega\rangle = \omega |\omega\rangle$$  \hspace{1cm} (3.59)

The key point is that the eigenvector’s direction in the inner product space is left unchanged by the action of the operator. An operator can have multiple eigenvectors, and the eigenvectors need not all be different.

One of the postulates of QM is that measurement of any classical variable yields only the eigenvalues of the corresponding quantum operator, with only the probability of obtaining any particular value known ahead of time, and that the act of measuring the physical quantity results in collapse of the state to the eigenstate corresponding to the measured eigenvalue.

It is therefore not surprising that we must study the problem of eigenvalues and eigenvectors in inner product spaces.

You have seen material of this type repeatedly, in your discussion of both normal modes and of quantum mechanics in Ph2/12. As usual, though, we will proceed methodically to ensure you understand the eigenvalue-eigenvector problem deeply.
Statement of the Problem

Given a linear operator $\Omega$. How do we find its eigenvalues and eigenvectors? We are asking for solutions to the linear equation ($I$ is the identity operator)

$$\Omega |v\rangle = \omega |v\rangle \iff (\Omega - I \omega) |v\rangle = |0\rangle \quad (3.60)$$

Solution of the Problem

You know from studying linear algebra that the above equation is only true if the determinant of any matrix representation of the operator on the left side vanishes:

$$|\Omega - I \omega| = 0 \quad (3.61)$$

This equation is termed the characteristic equation for $\Omega$.

Here we begin to get sloppy about the difference between equality $=$ and representation $\leftrightarrow$: a determinant only makes sense for a matrix representation, not for an operator, but we are using the symbol for the operator in the above equation. We could dream up some notation to distinguish between the operator and a matrix representation of it; for example, $\Omega$ and $[\Omega]$ or $\Omega$ and $\Omega$. This will become very tedious to carry along for the remainder of the course, though, so from here on we will have to rely on context to distinguish between an operator and its matrix representation.
To properly justify Equation 3.61, one must: 1) prove a general formula for the inverse of a matrix when a matrix representation is specified; 2) assume that the operator \( \Omega - I \omega \) is noninvertible so that \([\Omega - I \omega] \ket{v} = \ket{0}\) does not imply \(\ket{v} = \ket{0}\); and 3) use noninvertibility and the inversion formula to obtain \(\Omega - I \omega = 0\). See Shankar Appendix A.1, Equation A.1.7 and Theorem A.1.1.

The formula only can be written explicitly when a matrix representation is specified for \(\Omega\) and \(\ket{v}\), which is only possible when an orthonormal basis is specified. Let’s assume this has been given. Then we can write out the determinant. Since we have put no conditions on \(\Omega\), all we can say at this point is that the the resulting equation is a \(n\)th-order polynomial in \(\omega\) where \(n\) is the dimension of the space: the diagonal of \(\Omega - I \omega\) has one power of \(\omega\) in each element, and the determinant will include one term that is the product of all these elements, so there is at least one term in \(\omega^n\). So, the eigenvalues will be given by the solution to the polynomial equation

\[
p_n(\omega) = \sum_{m=0}^{n} c_m \omega^m = 0 \tag{3.62}
\]

The polynomial \(p_n\) is called the characteristic polynomial for the operator \(\Omega\). The fundamental theorem of algebra tells us it has \(n\) roots, some possibly complex. If the vector space’s field is complex, then these are valid eigenvalues; if the field were real, then we say that some of the roots do not exist. Thus, any linear operator in a vector space whose field is the complex numbers is guaranteed to have as many eigenvalues as the dimension of the vector space. Since the eigenvalues are independent of the basis and the matrix representation (Equation 3.60 is basis- and representation-independent), the characteristic polynomial must also be.
Once we have the eigenvalues, how do we find the eigenvectors?

Easy: for a particular eigenvalue $\omega_j$ and eigenvector $|\omega_j\rangle$, we have the equation

$$\left(\Omega - I \omega_j\right) |\omega_j\rangle = |0\rangle$$  \hspace{1cm} (3.63)

Since we explicitly know what the operator is — we know the elements of $\Omega$ and we know $\omega_j$ — all we need to do is solve for the elements of $|\omega_j\rangle$. Formally, though, because the determinant of the matrix on the left vanishes, we are not guaranteed a unique solution. What we end up with is $n - 1$ independent linear equations that determine $n - 1$ components of $|\omega_j\rangle$, leaving the overall normalization undetermined. The normalization of $|\omega_j\rangle$ is arbitrary since, if $|\omega_j\rangle$ is an eigenvector, then $\alpha|\omega_j\rangle$ will also be an eigenvector for any $\alpha$.

Of course, if our vector space has a real field, which may result in some of the eigenvalues not existing in the field, then the corresponding eigenvectors will also not exist because we would simply not be allowed to write Equation 3.63 for that eigenvalue.

In some cases, the above procedure will not yield the $n$ eigenvectors in that one will obtain $|\omega_j\rangle = |0\rangle$; this happens when there are degenerate (equal) eigenvalues. We can prove some explicit theorems about the existence of eigenvectors and the nature of the eigenvalues when the operators are Hermitian or unitary, which we will do below.
Example 3.31: In $\mathbb{C}^3$, rotation about the vector $(\hat{x} + \hat{y} + \hat{z})/\sqrt{3}$, which simply cyclically permutes the three unit vectors.

The matrix representation of this operator, which rotates $\hat{x} \rightarrow \hat{y}$, $\hat{y} \rightarrow \hat{z}$ and $\hat{z} \rightarrow \hat{x}$, is

$$A \leftrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$0 = |A - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 1 = 0$$

$$\omega_1 = 1 \quad \omega_2 = e^{2\pi i/3} \quad \omega_3 = e^{-2\pi i/3}$$

If we had assumed $\mathbb{R}^3$, we would say that two of the eigenvalues and the corresponding eigenvectors do not exist.

Let us find the eigenvectors for each case by calculating $A - \lambda I$ for each case and solving $(A - \lambda I)|v\rangle = |0\rangle$. 
\[ \omega_1 = 1: \]

\[
\begin{bmatrix}
-1 & 0 & 1 \\
1 & -1 & 0 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
v_{11} \\
v_{12} \\
v_{13}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \quad \Rightarrow \\
-v_{11} + v_{13} = 0 \\
v_{11} - v_{12} = 0 \\
v_{12} - v_{13} = 0
\]

\[ \Rightarrow \ket{\omega_1} \leftrightarrow \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

As expected, the normalization is not set because the three equations are not independent. The conventional choice is to normalize to 1, so in this case \( \alpha = 1/\sqrt{3} \). As one would expect, the vector corresponding to the axis of rotation has eigenvalue 1.
\[ \omega_2 = e^{\frac{2\pi i}{3}} : \]

\[
\begin{bmatrix}
  -e^{\frac{2\pi i}{3}} & 0 & 1 \\
  1 & -e^{\frac{2\pi i}{3}} & 0 \\
  0 & 1 & -e^{\frac{2\pi i}{3}}
\end{bmatrix}
\begin{bmatrix}
  v_{21} \\
  v_{22} \\
  v_{23}
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}
\implies
\begin{align*}
  -e^{\frac{2\pi i}{3}} v_{21} + v_{23} &= 0 \\
  v_{21} - e^{\frac{2\pi i}{3}} v_{22} &= 0 \\
  v_{22} - e^{\frac{2\pi i}{3}} v_{23} &= 0
\end{align*}
\]

\[
\implies |\omega_2 \rangle \leftrightarrow \alpha
\begin{bmatrix}
  1 \\
  e^{-\frac{2\pi i}{3}} \\
  e^{\frac{2\pi i}{3}}
\end{bmatrix}
\]

Since all the elements are unit modulus, we again may take \( \alpha = 1/\sqrt{3} \). Had we restricted to \( \mathbb{R}^3 \), we would have said this eigenvector did not exist (which makes sense, given that the eigenvalue would not have been in the scalar field of the vector space).
\( \omega_3 = e^{-\frac{2\pi i}{3}} : \)

\[
\begin{bmatrix}
-e^{-\frac{2\pi i}{3}} & 0 & 1 \\
1 & -e^{-\frac{2\pi i}{3}} & 0 \\
0 & 1 & -e^{-\frac{2\pi i}{3}}
\end{bmatrix}
\begin{bmatrix}
v_{31} \\
v_{32} \\
v_{33}
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\implies
-e^{-\frac{2\pi i}{3}} v_{31} + v_{33} = 0
\]

\[
\begin{align*}
v_{31} - e^{-\frac{2\pi i}{3}} v_{32} &= 0 \\
v_{32} - e^{-\frac{2\pi i}{3}} v_{33} &= 0
\end{align*}
\]

\[
\implies |\omega_3 \rangle \leftrightarrow \alpha
\begin{bmatrix}
1 \\
e^{\frac{2\pi i}{3}} \\
e^{-\frac{2\pi i}{3}}
\end{bmatrix}
\]

Again, we may take \( \alpha = 1/\sqrt{3} \), and, again, this eigenvector would be said to not exist if we had restricted to \( \mathbb{R}^3 \).