Lecture 35:
Coordinate Transformations:
Passive Coordinate Transformations

Date Revised: 2009/01/23
Date Given: 2009/01/21
Coordinate Transformations

A coordinate transformation is any relabeling of the points in space from set of axes $F$ with coordinates $q$ to a different set of axes $F'$ with coordinates $q'$, with $q' = q'(q, t)$. Some examples:

- **Mirror transformation:**
  \[ x' = -x \quad y' = y \quad z' = z \]

- **Rotation transformation about the z axis by $\theta$ (CCW):**
  \[ x' = x \cos \theta + y \sin \theta \quad y' = -x \sin \theta + y \cos \theta \quad z' = z \]

- **Translation:**
  \[ x' = x - a \quad y' = y \quad z' = z \]

In all of these, a new set of axes are being imposed on space and one chooses to label points relative to the new system rather than the old one. You should draw the old and new axes so you are clear on what is meant by each transformation.
Discrete vs. Continuous Transformations

A discrete coordinate transformation is one for which there is no continuous parameter that varies the effect of the transformation. A continuous coordinate transformation is therefore one that does depend on a continuous parameter. Continuous transformations can be differentiated with respect to the parameter.

Examples of discrete transformations include parity transformation (inversion of the coordinates through the origin) and mirror transformations (inversion through a plane; of course, one can vary the plane, but once one picks a plane, there is just one transformation) are a couple obvious ones.

Continuous transformations include spatial translations and rotations, where, even after one has picked a direction for the transformation (a translation direction or rotation axis), there is a continuous parameter: the translation distance or rotation angle in these cases.

Discrete transformations in general need to be treated on a case-by-case basis, whereas continuous transformation can be treated in a unified manner. The latter also result in conserved quantities, as we shall see.
Effect of Coordinate Transformations in Quantum Mechanics

A relabeling of the above type has no effect on the particle state $|\psi\rangle$ because it is an abstract object and makes no reference to a particular coordinate system. But, there are two ways in which the relabeling matters:

- First, because of the redefinition of the coordinate axes, the position and momentum basis change; there are new sets of basis states $\{|q'\rangle\}$ and $\{|p'_q\rangle\}$ that indicate states with well-defined position or momentum relative to the new axes rather than the old axes. The old position and momentum bases $\{|q\rangle\}$ and $\{|p_q\rangle\}$ still exist and the projections $\langle q | \psi \rangle$ and $\langle p_q | \psi \rangle$ are unchanged.

- Second, it affects the operators. We want the above new basis elements to be eigenvectors of position and momentum operators, so we require new position and momentum operators $\{Q'\}$ and $\{P_q'\}$, as these new basis elements will not be eigenvectors of the original position and momentum operators with the expected eigenvalues. These are not the same as the old operators $\{Q\}$ and $\{P_q\}$, meaning that the action of $\{Q'\}$ and $\{P_q'\}$ on states are different from that of $\{Q\}$ and $\{P_q\}$. The $\{Q\}$ and $\{P_q\}$ operators of course still exist and there is no change in their action.

Realize that the new bases live in the same Hilbert space as the old bases, and that the new and old operators both act there; the coordinate relabeling results in no new Hilbert space!
Passive Coordinate Transformations

We define the passive transformation operator $T_P$ to be the operator that maps from the original basis $\{|q\rangle\}$ to the new basis $\{|q'\rangle\}$, element-by-element:

$$e^{i\theta}|q'\rangle = T_P|q\rangle$$

where $\theta$ is real. That is, the passive transformation operator "transforms" the position-basis elements associated with the old coordinate system into position-basis elements associated with the new coordinate system. (We will define an active transformation below.) The $e^{i\theta}$ factor is allowed out front because it does not affect the norm of the state or the unitarity property derived below. In general, we will choose $\theta = 0$ so that the factor’s value is always unity, but that is an additional assumption that must be made; it need not always hold true.

Since we are just relabeling points in space, the dimensionality of the new basis must be the same as that of the old basis; that is, the mapping is one-to-one, or invertible. Assuming the new basis elements are normalized in the same way as the old ones (which is unaffected by the unity modulus factor), the transformation also preserves inner products between basis elements because the unit-modulus factor cancels out. Therefore, it preserves all inner products, and hence is unitary, $T_P^\dagger = T_P^{-1}$. 
We define position and momentum operators in the new basis by the requirement that their action in the new basis be the same as the corresponding old operators’ action in the old basis; this follows from their being position and momentum operators for the new coordinate system. So,

\[ \langle q_1' \mid Q' \mid q_2' \rangle = \langle q_1 \mid Q \mid q_2 \rangle \]

By the definition of the transformation operator’s action on states, \(|q'\rangle = T_P |q\rangle\), we also have \( \langle q_1' \mid Q' \mid q_2' \rangle = \langle q_1 \mid T_P^\dagger Q' T \mid q_2 \rangle \). Combining the two statements gives

\[ \langle q_1 \mid Q \mid q_2 \rangle = \langle q_1 \mid T_P^\dagger Q' T \mid q_2 \rangle \]

Since this relation holds for all \(|q_1\rangle\) and \(|q_2\rangle\), it therefore holds that

\[ Q' = T_P Q T_P^\dagger = T_P Q T_P^{-1} \]

The above proof carries through for the \(\{P_q\}\) also, and, in fact, for any operator. Because it holds for any pair of basis elements, it holds for any pair of states, giving

\[ \langle v' \mid O' \mid w' \rangle = \langle v \mid O \mid w \rangle \quad \iff \quad O' = T_P Q T_P^\dagger \] (12.1)
Coordinate Transformations (cont.)

We may thus think of the passive transformation operator as not only transforming the basis elements but also transforming any operator, with the new operator defined by the requirement that the action of the new operator in the new basis is the same as the old operator in the old basis.

Note that the passive transformation operator affects any basis, not just the position basis. All of our bases are defined by being eigenbases of some Hermitian operator ($Q$, $P_q$, etc.). If the transformation changes these operators, then their eigenbases will change, too. One obtains the transformed eigenbases by diagonalizing the transformed operators.

We note that, in general, in addition to the defining transformation relation Equation 12.1, it is also possible to rewrite the primed operators in terms of the unprimed operators simply because the primed observables are also observables in the unprimed coordinate system and hence must be functions of the $\{Q\}$ and $\{P_q\}$. The form of these latter relations will depend on the particular type of transformation, though.

Additionally, because we do not want to move the potentials and particles when we perform a coordinate transformation, we do not in general want to transform $H$ to $H'$. We will, though, write $H$ in terms of the $\{Q'\}$ and $\{P'\}$ operators by using this latter set of relations that write the primed operators in terms of the unprimed operators (and vice versa).
Example 12.1: Passive Mirror Transformation of a Particle in a Two-Dimensional Box

Consider a particle in the an eigenstate of a two-dimensional box of dimension $L_1 \times L_2$ whose corner is at the origin. Let the state be denoted by $|\psi_{ab}\rangle$. The energy is

$$E_{ab} = \frac{\hbar^2 \pi^2}{2 m} \left( \frac{a^2}{L_1^2} + \frac{b^2}{L_2^2} \right)$$

The position-basis representation is

$$\psi_{ab}(x, y) = \langle x, y | \psi_{ab} \rangle = \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{a \pi x}{L_1} \right) \sin \left( \frac{b \pi y}{L_2} \right)$$

(Note that, because the box’s corner, not center, is at the origin, the sines are the only allowed states. We do this to avoid having to carry around sines and cosines in this example.) and we would find the expectation values

$$\langle \psi_{ab} | X | \psi_{ab} \rangle = \frac{L_1}{2} \quad \langle \psi_{ab} | Y | \psi_{ab} \rangle = \frac{L_2}{2} \quad \langle \psi_{ab} | P_x | \psi_{ab} \rangle = 0 \quad \langle \psi_{ab} | P_y | \psi_{ab} \rangle = 0$$
Coordinate Transformations (cont.)

Now, perform the coordinate transformation $x' = y, y' = x$, which is a mirroring through the line $x = y$. The transformation is visualized in the figure below. We see clearly that the axes are relabeled but the potential has not moved in space; this is what we mean by a passive transformation.
The transformation gives $|x' = u, y' = v\rangle = T_P |x = u, y = v\rangle$ where $u$ and $v$ are just numbers, not tied to any coordinate system. We'll see below why we use the $u$ and $v$. Be sure you understand what is meant by this notation, in particular the ordering of $u$ and $v$: for example, a state along the $x$ axis, $|x = 1, y = 0\rangle$, gets mapped to a state along the $x'$ axis, $|x' = 1, y' = 0\rangle$. The transformation rules tell us that $|x' = u, y' = v\rangle = |x = v, y = u\rangle$. That is, when written in terms of unprimed coordinates, the state that $|x = u, y = v\rangle$ gets mapped to by $T_P$ is the state $|x = v, y = u\rangle$ in the unprimed coordinates. In the example, the state on the $x'$ axis $|x' = 1, y' = 0\rangle$ is equal to the state on the $y$ axis $|x = 0, y = 1\rangle$, and the state $|x' = 0, y' = 1\rangle$ on the $y'$ axis is equal to the state $|x = 1, y = 0\rangle$ on the $x$ axis. Make sure you have the above straight; it is confusing! These relationships are also indicated in the figure.

The state $\psi_{ab}$ stays unchanged, but it must now be written in terms of the basis elements in the new axes, the $\{|x', y'\rangle\}$. We denote this wavefunction by $\psi_{ab, q'}(x', y')$ and it is given by $\psi_{ab, q'}(x', y') = \langle x', y' | \psi_{ab} \rangle$. We need the extra $q'$ subscript to indicate that $\psi_{ab, q'}$ is a different function of its arguments than the old $\psi_{ab}(x, y)$, which we shall now denote by $\psi_{ab, q}(x, y)$. This is consistent with our generic notation of denoting a wavefunction in some particular representation (basis) with a subscript that specifies the representation: the $\{|x', y'\rangle\}$ and $\{|x, y\rangle\}$ are different bases, so the representation of $|\psi_{ab}\rangle$ in the two bases are different and need to be distinguished.
How do we calculate $\psi_{ab,q'}(x',y')$, explicitly? We recall that $|x' = u, y' = v\rangle = |x = v, y = u\rangle$. So, we have:

$$
\psi_{ab,q'}(x' = u, y' = v) = \langle x' = u, y' = v | \psi_{ab} \rangle \\
= \langle x = v, y = u | \psi_{ab} \rangle \\
= \psi_{ab,q}(x = v, y = u) \\
= \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{a \pi v}{L_1} \right) \sin \left( \frac{b \pi u}{L_2} \right)
$$

So, $\psi_{ab,q'}(x',y') = \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{b \pi x'}{L_2} \right) \sin \left( \frac{a \pi y'}{L_1} \right)$

We went through these specific numbers $u$ and $v$ to avoid confusion about when you replace $(x, y)$ by $(x', y')$ and when by $(y', x')$: $u$ and $v$ are just numbers, not tied to any coordinate system. We know that the state at $|x' = u, y' = v\rangle$ is the same state as $|x = v, y = u\rangle$, and knowing that lets us rewrite the wavefunction in terms of $x'$ and $y'$. The above form should be intuitively obvious, though; if the $x'$ axis lies along the $y$ axis and the $y'$ axis lies along the $x$ axis, and the state does not change, then $x'$ must take on the role of $y$ and $y'$ of $x$. 
What are the new operators? As we said, the formula $O' = T_P O T_P^\dagger$ is abstract. To figure it out, we resort to matrix elements. We require

$$\langle x'_1, y'_1 | X' | x'_2, y'_2 \rangle = \langle x_1, y_1 | X | x_2, y_2 \rangle$$

Again, it will be far clearer if we use specific numbers not tied to a coordinate system. So let's write the above as

$$\langle x' = u_1, y' = v_1 | X' | x' = u_2, y' = v_2 \rangle = \langle x = u_1, y = v_1 | X | x_2 = u_2, y_2 = v_2 \rangle$$

Note how the $u$'s and $v$'s are distributed: the state $|x' = u_1, y' = v_1 \rangle$ satisfies

$|x' = u_1, y' = v_1 \rangle = T_P |x = u_1, y = v_1 \rangle$ and also $|x'_1 = u_1, y'_1 = v_1 \rangle = |x = v_1, y = u_1 \rangle$. The matrix element on the right is

$$\langle x = u_1, y = v_1 | X | x = u_2, y = v_2 \rangle = u_1 \delta(u_1 - u_2) \delta(v_1 - v_2)$$
So our requirement becomes

\[ \langle x' = u_1, y' = v_1 | X' | x' = u_2, y' = v_2 \rangle = u_1 \delta(u_1 - u_2) \delta(v_1 - v_2) \]

which may now be rewritten as

\[ \langle x', y' | X' | x', y' \rangle = x'_1 \delta(x'_1 - x'_2) \delta(y'_1 - y'_2) \]

Thus, our operator \( X' \) does exactly what we expect it to do: it has as eigenvectors the states \( \{|x', y'\rangle\} \) with eigenvalues \( \{x'\} \). It therefore is the \( X' \) operator.

If you do not use the \( u' \)’s and \( v' \)’s when trying to prove the above statement, it is very easy to get confused and end up thinking that the \( X' \) operator pulls out the \( y' \) coordinate of \( |x', y'\rangle \) because you end up with \( x_1 \) instead of \( u_1 \) at the front of the expression, and the transformation says \( y'_1 = x_1 \), so one is tempted to replace the \( x_1 \) with \( y'_1 \) and think that one gets \( Y' \) instead of \( X' \). This problem occurs because it is unclear whether one should use the relation \( (x_1, y_1) \rightarrow (x'_1, y'_1) \) or the relation \( |x'_1, y'_1\rangle = |y_1, x_1\rangle \). By instead using these “numbers” \( (u_1, v_1) \) and \( (u_2, v_2) \), one doesn’t get into notational quandaries like this.
By analogy, we may also conclude

\[ \langle x'_1, y'_1 | Y' | x'_2, y'_2 \rangle = y'_1 \frac{d}{dx'_1} \delta(x'_1 - x'_2) \delta(y'_1 - y'_2) \]

\[ \langle x'_1, y'_1 | P'_x | x'_2, y'_2 \rangle = -i \hbar \frac{d}{dx'_1} \delta(x'_1 - x'_2) \delta(y'_1 - y'_2) \]

\[ \langle x'_1, y'_1 | P'_y | x'_2, y'_2 \rangle = -i \hbar \frac{d}{dy'_1} \delta(x'_1 - x'_2) \delta(y'_1 - y'_2) \]

That is, all the primed operators do exactly what we expect them to in terms of the primed coordinate system. In obtaining the above, we will have taken a derivative with respect to the “numbers” \( u_1 \) and \( v_1 \) because they are indeed still variables: the point of introducing them is that they are not notationally tied to a particular coordinate system, so they remove the confusion noted above.

We have defined the primed operators by requiring that their matrix elements in the primed basis be the same as the matrix elements of the unprimed operators between the unprimed basis elements that are transformed into the primed basis elements. Since the unprimed and primed basis elements are not equal, this is not a statement of equality between the unprimed operators and primed operators.
However, because the unprimed and primed basis elements describe the same Hilbert space and there are equality relations between them (as illustrated in the figure), we can calculate the matrix elements of the primed operators in the unprimed basis and compare these to the matrix elements of the unprimed operators in the unprimed basis to obtain a relation between the primed and unprimed operators (or conversely, we could calculate the matrix elements of the unprimed operators in the primed basis and compare to the matrix elements of the primed operators in the primed basis).

Explicitly:

\[
\langle x = u_1, y = v_1 | X' | x = u_2, y = v_2 \rangle = \langle x' = v_1, y' = u_1 | X' | x' = v_2, y' = u_2 \rangle = v_2 \langle x' = v_1, y' = u_1 | x' = v_2, y' = u_2 \rangle = v_2 \langle x = u_1, y = v_1 | x = u_2, y = v_2 \rangle = \langle x = u_1, y = v_1 | Y | x = u_2, y = v_2 \rangle \Rightarrow \langle x_1, y_1 | X' | x_2, y_2 \rangle = \langle x_1, y_1 | Y | x_2, y_2 \rangle
\]

We find that all the possible matrix elements between unprimed basis elements of \(X'\) and \(Y\) are equal, so they must be the same operator:

\[X' = Y\]
But we also have a relation between $X'$ and $X$. Let’s write both of these down:

\[ X' = T_P X \ T_P^\dagger \quad X' = Y \]

Interesting, eh? The former is true, regardless of the situation, but the latter is far more useful in trying to understand this specific example. Similar relations hold for $Y'$, $P_x'$, and $P_y'$:

\[ Y' = T_P Y \ T_P^\dagger \quad Y' = X \]
\[ P_x' = T_P P_x \ T_P^\dagger \quad P_x' = P_y \]
\[ P_y' = T_P P_y \ T_P^\dagger \quad P_y' = P_x \]

For $P_x'$ and $P_y'$, one establishes the relations in the right column by taking matrix elements in the momentum basis rather than the position basis because these operators are diagonal in the momentum basis but not in the position basis. One can of course establish relationships between primed and unprimed momentum basis elements as we have done for position basis elements.

Simple relations like these do not always hold: they are specific to our mirror transformation example. As we noted above, though, it is always possible to write the \{Q'\} and \{P_q'\} operators in terms of the \{Q\} and \{P_q\} operators.
As we explained above, we do not transform the Hamiltonian, we simply rewrite it in terms of primed operators. But, for the sake of pedagogy, let us explicitly transform the Hamiltonian and see what goes wrong:

\[
H' = T_P \left( \frac{P_x^2 + P_y^2}{2m} + V(X, Y) \right) T_P^\dagger = \frac{(P'_x)^2 + (P'_y)^2}{2m} + V(X', Y')
\]

where we can make the step by thinking of \( V \) in terms of a Taylor expansion and inserting \( T_P^\dagger T \) between each factor of \( X, Y, P_x, \) or \( P_y \). Nothing obviously wrong happens in the kinetic terms, but the potential term has \( X' \) as its first argument and \( Y' \) as its second argument: the \( L_1 \) dimension of the box is along \( x' \) and the \( L_2 \) dimension along \( y' \). This corresponds to mirroring the box, which we did not want to do!

With that out of the way, let us rewrite \( H \) in terms of \( X', Y', P'_x, \) and \( P'_y \) using the relations like \( X' = Y \) proved above:

\[
H = \frac{P_x^2 + P_y^2}{2m} + V(X, Y) = \frac{(P'_y)^2 + (P'_x)^2}{2m} + V(Y', X')
\]

\( X' \) and \( Y' \) are in the right arguments of \( V \) to have the \( L_1 \) side of the box along \( y' \) and the \( L_2 \) side along \( x' \).
The above rewriting of $H$ tells us two things: first, our prior rewriting of the eigenstate wavefunctions in the $\{|x', y'\rangle\}$ basis was correct: we expect the $x'$ coordinate to be linked up with $L_2$ and the $y'$ coordinate to be linked up with $L_1$, as we found earlier for $\psi_{ab,q'}(x', y')$. Second, the energies of the states are unchanged because the Hamiltonian is unchanged.

We may calculate the expectation values of the operators in two ways. The first method is to do the necessary integrals for, e.g., $\langle \psi_{ab} | X' | \psi_{ab} \rangle$, explicitly by inserting completeness relations for the new $\{|x', y'\rangle\}$ basis. The second method is to simply use the relations between the primed and unprimed operators (like $X' = Y$) that we proved above, making the whole exercise trivial:

$$\langle \psi_{ab} | X' | \psi_{ab} \rangle = \langle \psi_{ab} | Y | \psi_{ab} \rangle = \frac{L_2}{2}$$

$$\langle \psi_{ab} | Y' | \psi_{ab} \rangle = \langle \psi_{ab} | X | \psi_{ab} \rangle = \frac{L_1}{2}$$

$$\langle \psi_{ab} | P'_{x} | \psi_{ab} \rangle = \langle \psi_{ab} | P_{y} | \psi_{ab} \rangle = 0$$

$$\langle \psi_{ab} | P'_{y} | \psi_{ab} \rangle = \langle \psi_{ab} | P_{x} | \psi_{ab} \rangle = 0$$