Lecture 38:
Coordinate Transformations:
Examples of Passive vs. Active Coordinate Transformations
Symmetry Transformations

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Example 12.4: Active Mirror Transformation of a Particle in a Two-Dimensional Box

Let’s consider the active-transformation version of Example 12.1. The active version corresponds to inverting the box through the line $x = y$, putting the $L_1$ dimension along the $y$ axis and the $L_2$ dimension along $x$.

We define a new coordinate system by $x' = y$ and $y' = x$ in the same way as for the passive transformation. We have the transformed basis $\{|x', y'\rangle = T_A |x, y\rangle\}$ (we drop the unity-modulus factor again), transformed operators $\{Q' = T_A Q T_A^\dagger\}$ and $\{P_{q'} = T_A P_q T_A^\dagger\}$, and the transformed eigenstate $|\psi'_{ab}\rangle = T_A |\psi_{ab}\rangle$. As we noted earlier in our general discussion, the action of $T_A$ on basis elements and operators is identical to that of the corresponding $T_P$ operator.
Let's study the various wavefunctions we get out of these transformations. Remember first that, because of the unitarity of $T_A$, the projection of the transformed state onto the transformed basis is equal to the projection of the untransformed state onto the corresponding untransformed basis element:

$$\langle x' = u, y' = v | \psi'_{ab} \rangle = \langle x = u, y = v | T_A^\dagger T_A | \psi_{ab} \rangle = \langle x = u, y = v | \psi_{ab} \rangle$$

Note that the basis elements on the left and right side are related by $|x' = u, y' = v \rangle = T_A |x = u, y = v \rangle$, not by an equality! Another way of saying this is that we are not projecting onto $|x = u, y = v \rangle = |x' = v, y' = u \rangle$ on the left side of the equation.

Let's go through all the same things we did for the passive transformation example, starting with the wavefunction. The transformed basis elements $\{|x', y' \rangle\}$ are no different from those we found in the passive transformation case, so we still have

$$\psi_{ab,q}(x, y) = \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{a \pi x}{L_1} \right) \sin \left( \frac{b \pi y}{L_2} \right)$$

$$\psi_{ab,q'}(x', y') = \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{b \pi x'}{L_2} \right) \sin \left( \frac{a \pi y'}{L_1} \right)$$
The new twist is that we have the new state $|\psi'_{ab}\rangle$ that is defined by
$\langle x' = u, y' = v | \psi'_{ab}\rangle = \langle x = u, y = v | \psi_{ab}\rangle$. We may immediately write

$$
\psi'_{ab,q'}(x' = u, y' = v) = \psi_{ab}(x = u, y = v)
$$

$$
= \sqrt{\frac{4}{L_1L_2}} \sin \left( \frac{a \pi u}{L_1} \right) \sin \left( \frac{b \pi v}{L_2} \right)
$$

$$
= \sqrt{\frac{4}{L_1L_2}} \sin \left( \frac{a \pi x'}{L_1} \right) \sin \left( \frac{b \pi y'}{L_2} \right)
$$

or,

$$
\psi'_{ab,q'}(x', y') = \sqrt{\frac{4}{L_1L_2}} \sin \left( \frac{a \pi x'}{L_1} \right) \sin \left( \frac{b \pi y'}{L_2} \right)
$$

The functional dependence of $\psi'_{ab,q'}$ on $(x', y')$ is the same as that of $\psi_{ab,q}$ on $(x, y)$, as we have stated in our general discussion. Note of course that $\psi'_{ab,q'}(x', y') \neq \psi_{ab,q'}(x', y')$ because $|\psi'\rangle$ is a different state than $|\psi\rangle$ and hence their projections onto the same $\{|x', y'\rangle\}$ basis are different.
Let's look at the last thing to consider, $\psi'_{ab,q} = \langle x = u, y = v \mid \psi' \rangle$:

$$\langle x = u, y = v \mid \psi' \rangle = \langle x' = v, y' = u \mid \psi' \rangle$$

$$= \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{a \pi v}{L_1} \right) \sin \left( \frac{b \pi u}{L_2} \right)$$

$$\implies \psi'_{ab,q}(x, y) = \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{b \pi x}{L_2} \right) \sin \left( \frac{a \pi y}{L_1} \right)$$

Thus, we see a relationship between the active and passive transformations:

**Passive:** $\psi_{ab,q}'(x', y') = \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{b \pi x'}{L_2} \right) \sin \left( \frac{a \pi y'}{L_1} \right)$

**Active:** $\psi'_{ab,q}(x, y) = \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{b \pi x}{L_2} \right) \sin \left( \frac{a \pi y}{L_1} \right)$

We thus have that the functional dependence of the transformed-basis wavefunction of the untransformed state $\psi_{ab,q}'$ on the transformed coordinates is the same as that of the untransformed-basis wavefunction on the untransformed coordinates $\psi'_{ab,q}$. This is a surprising and unusual result, though one for which there is a good reason.
The above wavefunctions are the two projections \( \langle x = u, y = v \mid \psi'_{ab} \rangle \) and \( \langle x' = u, y' = v \mid \psi_{ab} \rangle \). (We again use the dummy variables \((u, v)\) to avoid confusion about what arguments we need to consider.) In general, they are not related to each other because they may be written as

\[
\langle x = u, y = v \mid \psi'_{ab} \rangle = \langle x = u, y = v \mid T_A \psi_{ab} \rangle \\
\langle x' = u, y' = v \mid \psi_{ab} \rangle = \langle x = u, y = v \mid T_A^\dagger \psi_{ab} \rangle = \langle x = u, y = v \mid T_P^\dagger \psi_{ab} \rangle
\]

However, in this specific case of a mirror transformation, they are related because \( T_A^2 = I \) and thus \( T_A^\dagger = T_A = T_P = T_P^\dagger \). As we noted in our general discussion, the generic result will be that the above kind of relation holds when the passive and active transformations are related by \( T_P = T_A^\dagger = T_A^{-1} \).
Recall that the passively and actively transformed operators will in general be equal when $T_P = T_A$, but in this case they will also be equal when $T_P = T_A^{-1}$ because $T_A^2 = I$. That is, we have

$$T_P O T_P^\dagger = T_A O T_A^\dagger$$  in general

$$T_P O T_P^\dagger = T_A^{-1} O \left( T_A^{-1} \right)^\dagger = T_A^\dagger O T_A$$  for this particular case

In particular, the relation between transformed and untransformed operators will not depend on whether we consider $T_P = T_A$ or $T_P = T_A^{-1}$: we always get $X' = Y$, $Y' = X$, $P'_x = P_y$ and $P'_y = P_x$. 


Let’s now consider the Hamiltonian operator. We have from before

\[ H(Q', P_{q'}) = \frac{[P_x']^2 + [P_y']^2}{2m} + V(Y', X') \]

For the active transformation, we have

\[ H'(Q, P_q) = \frac{[P_x]^2 + [P_y]^2}{2m} + V(Y, X) \]

(Note: when we write \( H \) as a function of \( Q' \) and \( P_{q'} \), we intend “take \( H(Q, P_q) \) and use the relationships between \( Q, P_q \) and \( Q', P_{q'} \) to substitute for \( Q \) and \( P_{q'} \);” we do not mean “replace \( Q, P_q \) in \( H \) with \( Q', P_{q'} \) directly.” The same kind of statement holds for writing \( H' \) as a function of \( Q \) and \( P_q \). However, when we write \( V \), we do mean to treat it as a simple function of its arguments.) We see that the dependence of the untransformed Hamiltonian and the transformed operators is the same as that of the transformed Hamiltonian on the untransformed operators. This is again a special case resulting from the fact \( T_P^2 = I \) and \( T_A^2 = I \). In general, the above functional dependences will not be the same and one will have to require instead \( T_P = T_A^{-1} \) in order for them to match up.
One last comment on eigenstates. We know from our general discussion that \( |\psi'_{ab}\rangle = T_A |\psi_{ab}\rangle \) is an eigenstate of \( H' \) with energy
\[
E_{ab} = \frac{\hbar^2}{2m} \left( \frac{a^2}{L_1^2} + \frac{b^2}{L_2^2} \right) / 2. 
\]
It is fairly obvious that \( |\psi_{ab}\rangle \) is not an eigenstate of \( H' \) and \( |\psi'_{ab}\rangle \) is not an eigenstate of \( H \) because the functional dependences of the corresponding wavefunctions on the coordinates do not match up with the potential in the corresponding Hamiltonian. So, though \( \psi_{ab,q}(x', y') \) may depend on its arguments in the same way that \( \psi'_{ab,q}(x, y) \) depends on its arguments, these are not somehow the same state. They are different states simply because \( |\psi_{ab}\rangle \not= |\psi'_{ab}\rangle \) no matter what basis one projects them on to, and of course one can only test equality between two states by projecting them onto the same basis. We can see from the work above that \( \langle x, y | \psi'_{ab} \rangle \not= \langle x, y | \psi_{ab} \rangle \) and \( \langle x', y' | \psi'_{ab} \rangle \not= \langle x', y' | \psi_{ab} \rangle \).
Active Coordinate Transformations (cont.)

Example 12.5: Active Rotation Transformation of a Particle in a Two-Dimensional Box

Now, let’s go through the same rigamarole for the active rotation transformation of the particle in a box. We’ll go pretty quickly since we have a lot of experience in doing these transformations by now.

Let’s write down the four possible wavefunctions:

\[
\psi_{ab,q}(x, y) = \langle x, y | \psi_{ab} \rangle = \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{a \pi x}{L_1} \right) \sin \left( \frac{b \pi y}{L_2} \right)
\]

\[
\psi_{ab,q'}(x', y') = \langle x', y' | \psi_{ab} \rangle = \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{a \pi (x' c_\theta - y' s_\theta)}{L_1} \right) \sin \left( \frac{b \pi (x' s_\theta + y' c_\theta)}{L_2} \right)
\]

\[
\psi'_{ab,q}(x, y) = \langle x, y | \psi'_a \rangle = \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{a \pi (x c_\theta + y s_\theta)}{L_1} \right) \sin \left( \frac{b \pi (-x s_\theta + y c_\theta)}{L_2} \right)
\]

\[
\psi'_{ab,q'}(x', y') = \langle x', y' | \psi'_{ab} \rangle = \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{a \pi x'}{L_1} \right) \sin \left( \frac{b \pi y'}{L_2} \right)
\]
Let’s explicitly derive the third one because it is the only interesting one we have not dealt with before:

$$
\psi'_{ab, q}(x = u, y = v) = \langle x = u, y = v | \psi'_{ab} \rangle = \langle x' = u c_\theta + v s_\theta, y' = -u s_\theta + v c_\theta | \psi'_{ab} \rangle = \psi'_{ab, q'}(x' = u c_\theta + v s_\theta, y' = -u s_\theta + v c_\theta) = \sqrt{\frac{4}{L_1 L_2}} \sin \left( \frac{a \pi (u c_\theta + v s_\theta)}{L_1} \right) \sin \left( \frac{b \pi (-u s_\theta + v c_\theta)}{L_2} \right)
$$

where the equality in the first line between untransformed and transformed basis elements is the same equality we justified in Example 12.2.

Going back to the four wavefunctions, we clearly see that the dependence of the untransformed wavefunction on the transformed coordinates, $$\psi'_{ab, q'}(x', y')$$, is the same as that of the transformed wavefunction on the untransformed coordinates, $$\psi'_{ab, q}(x, y)$$, if and only if we take the angle for the latter active transformation to be the opposite of the angle for the former passive transformation. That is, as we said for the general case, we require $$T_P = T_A^{-1}$$ in order to get these functional dependences to match up.
As we noted earlier, when we take $T_P = T_A$, the transformed operators are the same regardless of whether the transformation is active or passive. We remind you that these relations are

\[
\begin{align*}
X' &= X c_\theta + Y s_\theta \\
Y' &= -X s_\theta + Y c_\theta \\
P'_x &= P_x c_\theta + P_y s_\theta \\
P'_y &= -P_x s_\theta + P_y c_\theta
\end{align*}
\]

Of course, if we want the wavefunction functional dependences to match up, then the passively and actively transformed operators will not be equal because they will have different signs on $\theta$. 
Let’s look at the relation between the untransformed and transformed Hamiltonians:

\[
H(Q, P_q) = \frac{[P_x]^2 + [P_y]^2}{2m} + V(X, Y)
\]

\[
H(Q', P_q') = \frac{[P_x']^2 + [P_y']^2}{2m} + V'(X'c_\theta - Y's_\theta, X's_\theta + Y'c_\theta)
\]

\[
H'(Q, P_q) = \frac{[P_x]^2 + [P_y]^2}{2m} + V(Xc_\theta + Ys_\theta, -Xs_\theta + Yc_\theta)
\]

\[
H(Q', P_q') = \frac{[P_x']^2 + [P_y']^2}{2m} + V'(X', Y')
\]

We have obtained \( H'(Q', P_q') \) by \( H' = T_AHT_A^\dagger \), which we calculate by inserting \( T_A^\dagger T_A \) between every power of untransformed operators and thus transforming them. The result should be intuitively clear, though, because it just consists of replacing every operator \( O \) in \( H \) with its transformed version \( O' \). We see that the functional dependence of \( H \) on \( X' \) and \( Y' \) only matches up with the functional dependence of \( H' \) on \( X \) and \( Y \) when the sign of \( \theta \) for the active and passive transformations are opposite, as we expect from our general discussion.
Passive and Active Transformations Use the Same Operator

Note that we now no longer need to distinguish between active and passive transformations at an operator level, as we saw that they act on the bases in the same way. The distinction between passive and active transformations is more a matter of which things one looks at: in the passive case, one cares about writing the untransformed state in terms of the transformed basis elements and the untransformed Hamiltonian in terms of the transformed operators; in the active case, one focused on writing the transformed state in terms of the untransformed basis elements and the transformed Hamiltonian in terms of the untransformed operators. When one compares the two, one sees that the passive and active transformations corresponding to a given transformation operator $T$ do different things. But this is a matter of interpretation, not a matter of a distinction between passive and active transformations at an operator level.