Physics 125a – Problem Set 2 – Due Oct 15, 2007
Solutions

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v. 2: Minor correction in (1d) to get normalization of P_1 and P_2 correct.
v. 3: Errors in entries for unitary matrix U in Problem 4 corrected – miscopied the eigenvectors into the unitary matrix.

Problem 1

(a) We require $K^\dagger = K$ and derive the consequences:

$$K^\dagger = (|v\rangle\langle w|)^\dagger = |w\rangle\langle v| \tag{1}$$

If $|v\rangle = (v_1, \cdots, v_n)^t$, $|w\rangle = (w_1, \cdots, w_n)^t$, then we have

$$K_{ij} = v_i w_j^* = (K^\dagger)_{ij} = w_i v_j^* \tag{2}$$

This can be rearranged to yield

$$\frac{v_i}{w_i} = \frac{v_j^*}{w_j^*} = \lambda \tag{3}$$

where λ is some constant. We see that λ must be real since taking complex conjugate does not change the equation. (note that i, j are completely arbitrary) Therefore, $|v\rangle = \lambda|w\rangle$ where λ is a real number.

(b) Let's calculate K^2.

$$K^2 = |w\rangle\langle v|w\rangle\langle v| = \langle v|w\rangle|w\rangle\langle v| \tag{4}$$

In order to make this equal to K, we must have $\langle v|w\rangle = 1$.

(c) No, the conditions are not the same. If condition (a) is satisfied, we can still have $\langle v|w\rangle \neq 1$, because these vectors might not be normalized to have unit norm. Conversely, if $\langle v|w\rangle = 1$, we can still make $|v\rangle \neq |w\rangle$ by choosing suitable norm of these vectors if they're not orthogonal. If one requires both conditions, then one obtaines that $|v\rangle = |w\rangle$ and $\langle v|v\rangle = 1$.

(d) Choose $P_1 = |v\rangle\langle v|/|v|^2$, and $P_2 = |w\rangle\langle w|/|w|^2$. Then we have

$$P_1 P_2 = \frac{1}{|v|^2|w|^2}|v\rangle\langle v|w\rangle\langle w| = \lambda^{-1}K \tag{5}$$

where $\lambda = |v|^2|w|^2/\langle v|w\rangle$.

Problem 2

$$\Omega = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} \tag{6}$$

(a) To find know the eigenvalues of this matrix, we solve $\det(\Omega - \omega I) = 0$ for ω.

$$\det\begin{pmatrix} 1-\omega & 0 & 0 \\ 0 & \frac{3}{2}-\omega & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2}-\omega \end{pmatrix} = 0 \tag{7}$$

$$(1-\omega)\left(\left(\frac{3}{2}-\omega\right)^2 - \frac{1}{4}\right) = (1-\omega)(\omega-1)(\omega-2) = 0 \tag{8}$$

Thus, we have $\omega_1 = \omega_2 = 1; \omega_3 = 2$.

(b) To find the eigenvector $|\omega_3\rangle$ with eigenvalue $\omega_3 = 2$, we must solve

$$(\Omega - \omega_3 I)|\omega_3\rangle = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \tag{9}$$

We obtain $x = 0, y + z = 0$. Thus, the eigenvector can be written as

$$|\omega_3\rangle = N_3\begin{pmatrix} 0 \\ a \\ -a \end{pmatrix} = \frac{1}{\sqrt{2a^2}}\begin{pmatrix} 0 \\ a \\ -a \end{pmatrix} \tag{10}$$

where N_3 is determined to make $\langle\omega_3|\omega_3\rangle = 1$.

(c)

$$(\Omega - \omega_1 I)|\omega_1\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \tag{11}$$

Here, we find $y = z$, and x is arbitrary. Thus, the eigenvector is

$$|\omega_1\rangle = N_1\begin{pmatrix} b \\ c \\ c \end{pmatrix} = \frac{1}{\sqrt{b^2 + 2c^2}}\begin{pmatrix} b \\ c \\ -c \end{pmatrix} \tag{12}$$

Problem 3

(a) Any Hermitian or unitary matrix can be diagonalized by a suitable unitary transformation. We did not really prove this explicitly for unitary matrices in the notes, but we did prove that the eigenvalues of a unitary matrix are all unit modulus and the eigenvectors form an orthonormal set, so it follows immediately that we can construct a unitary matrix U to diagonalize a unitary matrix Ω from the eigenvectors of Ω just as we did for Hermitian matrices. Thus we can write $\Omega = U^\dagger D U$ where D is the diagonalized matrix of Ω. Then,

$$\det\Omega = \det(U^\dagger D U) = \det U^\dagger \det D \det U = \det D = \prod_{i=1}^{n} \omega_i \tag{13}$$

since $\det U = \det U^\dagger = 1$ and $D = \mathrm{diag}(\omega_1, \cdots, \omega_n)$.

(b) As (a), we can easily see

$$\mathrm{tr}\Omega = \mathrm{tr}(U^\dagger D U) = \mathrm{tr}(U U^\dagger D) = \mathrm{tr}(D) = \sum_{i=1}^{n} \omega_i \tag{14}$$

using the cyclic property of the trace $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$.

Problem 4

First, let's calculate the eigenvalues of both matrices.

$$\det(\Omega - \omega I) = \det \begin{pmatrix} 1-\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & 1-\omega \end{pmatrix} = \omega^2(2-\omega) = 0 \tag{15}$$

therefore the eigenvalues of Ω are $0, 2$, and 0 and are degenerate.

$$\det(\Lambda - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & 2-\lambda \end{pmatrix} = (\lambda+1)(\lambda-2)(\lambda-3) = 0 \tag{16}$$

Thus, we have eigenvalues $-1, 2, 3$ for Λ.

To diagonalize these matrices, we need three orthonormal eigenvectors to form the unitary transformation matrix. Although you can form them by using Gram-Schmidt orthogonalization process with Ω, its much easier to get the unitary matrix through matrix Λ. Moreover, we must get simultaneous eigenvectors of Ω and Λ. This is automatically guaranteed if we get non-degenerate eigenvectors. So let's calculate the eigenvectors of Λ. It can be easily shown that

$$|-1\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \qquad |2\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \qquad |3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{17}$$

From this, we can construct the unitary matrix U.

$$U = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \tag{18}$$

And this gives

$$U^\dagger \Lambda U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad U^\dagger \Omega U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{19}$$

Both matrices are diagonalized by the unitary transformation U.

Problem 5

Let's follow the procedure outlined in the text. First, let's write the equation of motion in matrix form.

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -2k/m & k/m \\ k/m & -2k/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \Lambda|x\rangle \tag{20}$$

Here $|x\rangle$ denotes $(x_1, x_2)^t$. Now we can diagonalize Λ and find eigenvalues and corresponding eigenvectors. The eigenvalues are $-k/m, -3k/m$, and eigenvectors are

$$|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{21}$$

Thus, the unitary transformation matrix is given by

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{22}$$

The diagonalized form of equation of motion can be written as follows.

$$R|\ddot{x}\rangle = (R\Lambda R^\dagger)R|x\rangle \tag{23}$$

$$|\ddot{v}\rangle = \begin{pmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_2^2 \end{pmatrix} |v\rangle \tag{24}$$

where $\omega_1^2 = k/m, \omega_2^2 = 3k/m$ and $|v\rangle = R|x\rangle$. Let $|v\rangle = (v_1, v_2)^t$. Then, we have equations for two simple harmonic oscillators with angular frequencies given by ω_1 and ω_2.

$$\ddot{v}_1 = -\omega_1^2 v_1 \tag{25}$$
$$\ddot{v}_2 = -\omega_2^2 v_2 \tag{26}$$

This can be easily solved to find

$$v_1 = v_1(t=0)\cos\omega_1 t + \frac{\dot{v}_1(t=0)}{\omega_1}\sin\omega_1 t \tag{27}$$
$$v_2 = v_2(t=0)\cos\omega_2 t + \frac{\dot{v}_2(t=0)}{\omega_2}\sin\omega_2 t \tag{28}$$

which we can write along the lines of what was done in the notes as

$$\begin{aligned} |v(t)\rangle &= \left[\langle 1|y(t=0)\rangle\cos\omega_1 t + \frac{1}{\omega_1}\langle 1|\dot{y}(t=0)\rangle\sin\omega_1 t\right]|1\rangle \qquad (29) \\ &\quad + \left[\langle 2|y(t=0)\rangle\cos\omega_2 t + \frac{1}{\omega_2}\langle 2|\dot{y}(t=0)\rangle\sin\omega_2 t\right]|2\rangle \\ &= \Big[\cos\omega_1 t|1\rangle\langle 1| + \cos\omega_2 t|2\rangle\langle 2|\Big]|v(t=0)\rangle \qquad (30) \\ &\quad + \left[\frac{1}{\omega_1}\sin\omega_1 t|1\rangle\langle 1| + \frac{1}{\omega_2}\sin\omega_2 t|2\rangle\langle 2|\right]|\dot{v}(t=0)\rangle \qquad (31) \\ &\equiv U_1(t)|v(t=0)\rangle + U_2(t)|\dot{v}(t=0)\rangle \end{aligned}$$

which defines two unitary time evolution operators $U_1(t)$ and $U_2(t)$. Now, we would like to obtain $|x(t)\rangle$ in terms of $|x(t=0)\rangle$ and $|\dot{x}(t=0)\rangle$, so we use our unitary transformation matrix:

$$|x(t)\rangle = R^\dagger |v(t)\rangle \qquad |v(t=0)\rangle = R|x(t=0)\rangle \tag{32}$$

$$|\dot{x}(t)\rangle = R^\dagger |\dot{v}(t)\rangle \qquad |\dot{v}(t=0)\rangle = R|\dot{x}(t=0)\rangle \tag{33}$$

(Note that R is not time-dependent!) This lets us rewrite

$$\begin{aligned}
|x(t)\rangle &= R^\dagger |v(t)\rangle && (34)\\
&= R^\dagger U_1(t)|v(t=0)\rangle + R^\dagger U_2(t)|\dot{v}(t=0)\rangle && (35)\\
&= R^\dagger U_1(t) R|x(t=0)\rangle + R^\dagger U_2(t) R|\dot{x}(t=0)\rangle && (36)\\
&= R^\dagger \Big[\cos\omega_1 t|1\rangle\langle 1| + \cos\omega_2 t|2\rangle\langle 2|\Big] R|x(t=0)\rangle && (37)\\
&\quad + R^\dagger \left[\frac{1}{\omega_1}\sin\omega_1 t|1\rangle\langle 1| + \frac{1}{\omega_2}\sin\omega_2 t|2\rangle\langle 2|\right] R|\dot{x}(t=0)\rangle && (38)
\end{aligned}$$

We may find $|\dot{x}(t)\rangle$ by simply taking the time derivative, recognizing that R is constant:

$$\begin{aligned}
|\dot{x}(t)\rangle &= R^\dagger |\dot{v}(t)\rangle && (39)\\
&= -R^\dagger \Big[\omega_1 \sin\omega_1 t|1\rangle\langle 1| + \omega_2 \sin\omega_2 t|2\rangle\langle 2|\Big] R|x(t=0)\rangle && (40)\\
&\quad + R^\dagger \Big[\cos\omega_1 t|1\rangle\langle 1| + \cos\omega_2 t|2\rangle\langle 2|\Big] R|\dot{x}(t=0)\rangle && (41)
\end{aligned}$$

The reason that the example with the initial velocities set to zero is a better analogy to the Schrödinger Equation than the full example we just did is that Schrödinger Equation is a first-order differential equation, so there would be no terms including the initial velocities as we found above; Newton's second law is a second-order differential equation, so the full solution requires both initial positions and velocities.