Physics 125a – Problem Set 3 – Due Oct 22, 2007
Solutions

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Problem 1

Let $|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then, we can write the operators P and R in matrix representation.

$$P = \begin{bmatrix} a & c \\ c^* & b \end{bmatrix} \tag{1}$$

$$R = \begin{bmatrix} A & w \\ w^* & B \end{bmatrix} \tag{2}$$

with a, b, A, and B real numbers. Then, we can directly substitute P and R into the matrix element to determine the parameters.

$$\langle 1|P|1\rangle = \frac{1}{2} \Rightarrow a = \frac{1}{2}$$
$$\langle 1|P^2|1\rangle = \frac{1}{4} \Rightarrow |c|^2 = 0 \to c = 0$$
$$\Rightarrow P = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & b \end{bmatrix} \text{with a free parameter } b \tag{3}$$

Similarly,

$$\langle 1|R|1\rangle = 1 \Rightarrow A = 1$$
$$\langle 1|R^2|1\rangle = \frac{5}{4} \Rightarrow |w|^2 = \frac{1}{4} \to w = \frac{1}{2}e^{i\theta}$$
$$\langle 1|R^3|1\rangle = \frac{7}{4} \Rightarrow B = 1$$
$$\Rightarrow R = \begin{bmatrix} 1 & \frac{1}{2}e^{i\theta} \\ \frac{1}{2}e^{-i\theta} & 1 \end{bmatrix} \text{with a free parameter } \theta \tag{4}$$

Then, we can solve the eigenvalue eqautions to get eigenvalues

$$\text{For P,} \begin{vmatrix} \frac{1}{2} - \lambda & 0 \\ 0 & b - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = b, \frac{1}{2} \tag{5}$$

$$\text{For R,} \begin{vmatrix} 1 - \lambda & \frac{1}{2}e^{i\theta} \\ \frac{1}{2}e^{-i\theta} & 1 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = \frac{1}{2}, \frac{3}{2} \tag{6}$$

Problem 2

(a)

$$\Gamma^\dagger = \sum_{j=1}^{n} E_0^* |j\rangle\langle j| = \sum_{j=1}^{n} E_0 |j\rangle\langle j| = \Gamma \tag{7}$$

$$\Delta^\dagger = \sum_{j=1}^{n} W^* [|j+1\rangle\langle j| + |j\rangle\langle j+1|] = \sum_{j=1}^{n} W [|j+1\rangle\langle j| + |j\rangle\langle j+1|] = \Delta \tag{8}$$

$$\Omega^\dagger = (\Gamma + \Delta)^\dagger = \Gamma^\dagger + \Delta^\dagger = \Gamma + \Delta = \Gamma \tag{9}$$

So Γ, Δ, and Ω are Hermitian. If one of the two terms were missing in the definition of Δ, we would have

$$\Delta^\dagger = \sum_{j=1}^{n} W [|j+1\rangle\langle j|] \neq \Delta \tag{10}$$

and Δ would not be Hermitian.

(b) There are two ways to do this.

(1) The easiest way is that we can see that in matrix representation, Γ is proportional to Identity matrix. Thus

$$[\Gamma, \Delta] = E_0[I, \Delta] = 0 \tag{11}$$

(2) Here I use the Dirac notations.

$$\begin{aligned}\Gamma\Delta &= \left(\sum_{j=1}^{n} E_0 |j\rangle\langle j|\right)\left(\sum_{j'=1}^{n} W[|j'\rangle\langle j'+1| + |j'+1\rangle\langle j'|]\right) \\ &= \sum_{j=1}^{n} E_0 W |j\rangle\langle j+1| + \sum_{j'=1}^{n} E_0 W |j'+1\rangle\langle j'| \\ &= \sum_{j=1}^{n} E_0 W \left(|j\rangle\langle j+1| + |j+1\rangle\langle j|\right)\end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned}\Delta\Gamma &= \left(\sum_{j=1}^{n} W[|j\rangle\langle j+1| + |j+1\rangle\langle j|]\right)\left(\sum_{j'=1}^{n} E_0 |j'\rangle\langle j'|\right) \\ &= \sum_{j=1}^{n} E_0 W |j\rangle\langle j+1| + \sum_{j=1}^{n} E_0 W |j+1\rangle\langle j| \\ &= \sum_{j}^{n} E_0 W \left(|j\rangle\langle j+1| + |j+1\rangle\langle j|\right)\end{aligned} \tag{13}$$

Then we can see $[\Gamma, \Delta] = 0$

(c) According to the problem, we make a guess about the form of eigenvectors, and now we directly write down the eigenvalue equation

$$\begin{pmatrix} E_0-\omega_l & w & \dots & \dots & \dots & \dots & \dots & w \\ w & E_0-\omega_l & w & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & w & E_0-\omega_l & w & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ w & \dots & \dots & \dots & \dots & \dots & w & E_0-\omega_l \end{pmatrix} \begin{pmatrix} e^{i\theta_l} \\ e^{2i\theta_l} \\ \dots \\ \dots \\ e^{ik\theta_l} \\ \dots \\ \dots \\ e^{in\theta_l} \end{pmatrix} = 0 \tag{14}$$

Let's consider the kth row of the characteristic equation, which gives us

$$\begin{aligned} & we^{i(k-1)\theta_l} + (E_0-\omega_l)e^{ik\theta_l} + \omega_l e^{i((k+1)\theta_l} = 0 \\ & \Rightarrow E_0 - \omega_l + 2w\cos(\theta_l) = 0 \\ & \Rightarrow \omega_l = E_0 + 2w\cos(\theta_l) \end{aligned} \tag{15}$$

Thus, we can see that the eigenvectors we guess actually work. So, to sum up, the eigenvalues are given above and eigenvectors are

$$\nu_l = \frac{1}{\sqrt{n}} \begin{pmatrix} e^{i\theta_l} \\ e^{i2\theta_l} \\ \dots \\ \dots \\ e^{in\theta_l} \end{pmatrix} \tag{16}$$

In order to see clearly how ω_l depends on θ_l, let's sketch a graph of the eigenvalue ω_l as a function of θ_l with $l = 1, \dots, n$.

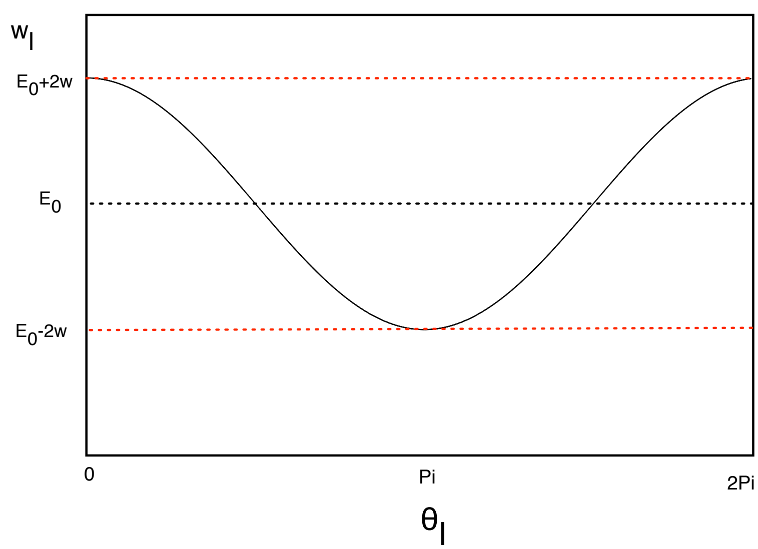

Figure 1: ω_l vs θ_l

Problem 3

In the eigenbasis of the Hermitian operator Ω, the matrix representation of Ω is

$$\Omega = \begin{pmatrix} \omega_1 & 0 & 0 & \dots \\ 0 & \omega_2 & 0 & \dots \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \dots & \omega_n \end{pmatrix} \tag{17}$$

Thus, it's easy to see that

$$\begin{aligned} &\sum_{n=0}^{\infty} \Omega^n \\ &= \begin{pmatrix} 1+\omega_1+\omega_1^2+\dots & 0 & 0 & \dots \\ 0 & 1+\omega_2+\omega_2^2+\dots & 0 & \dots \\ \multicolumn{4}{c}{\dotfill} \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \dots & 1+\omega_n+\omega_n^2+\dots \end{pmatrix} &(18) \\ &= \begin{pmatrix} (1-\omega_1)^{-1} & 0 & 0 & \dots \\ 0 & (1-\omega_2)^{-1} & 0 & \dots \\ \multicolumn{4}{c}{\dotfill} \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \dots & (1-\omega_n)^{-1} \end{pmatrix} \equiv (1-\Omega)^{-1} &(19) \end{aligned}$$

Thus, as far as convergence condition is concerned, each of the eigenvalue should be less than 1. Which means

$$\omega_i < 1 \quad \text{for } i = 1, \dots, n \tag{20}$$

Problem 4

(a) (1) In H's eigenbasis. We can see that

$$\begin{aligned} &\langle i|U^\dagger U|j\rangle = e^{i(\omega_j-\omega_i)}\langle i|j\rangle = \delta_{ij} \\ &\Rightarrow U^\dagger U = 1 \end{aligned} \tag{21}$$

Thus, $U = e^{iH}$ is unitary.

(2) Let's directly expand U and $U^\dagger$ using Taylor expansion

$$\begin{aligned} &U^\dagger U = e^{-iH}e^{iH} \\ &= \left[1 - iH - \frac{H^2}{2!} + i\frac{H^3}{3!} + \dots\right]\left[1 + iH - \frac{H^2}{2!} - i\frac{H^3}{3!} + \dots\dots\right] \\ &= \left[1 + iH - \frac{H^2}{2} - i\frac{H^3}{3!} + \frac{H^4}{4!} + i\frac{H^5}{5!} + \dots\dots\right] \\ &+ \left[-iH + H^2 + i\frac{H^3}{2} - \frac{H^4}{3!} - i\frac{H^5}{4!} + \dots\dots\right] \\ &+ \left[-\frac{H^2}{2} - i\frac{H^3}{2} + \frac{H^4}{2\times 2} + i\frac{H^5}{2\times 3!} + \dots\dots\right] \\ &+ \left[i\frac{H^3}{3!} - \frac{H^4}{3!} - i\frac{H^5}{2\times 3!} + \dots\dots\right] \\ &+ \left[\frac{H^4}{4!} + i\frac{H^5}{4!} - i\frac{H^5}{5!} + \dots\dots\right] \\ &= 1 \end{aligned} \tag{22}$$

Thus, U is unitary.

(b) In H's eigenbasis, we can write

$$iH = \begin{pmatrix} iE_1 & 0 & 0 & \dots \\ 0 & iE_2 & 0 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & iE_n \end{pmatrix} \tag{23}$$

Thus, we can know that

$$(iH)^n = \begin{pmatrix} (iE_1)^n & 0 & 0 & \dots \\ 0 & (iE_2)^n & 0 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (iE_n)^n \end{pmatrix} \tag{24}$$

Then, we can write U as

$$U = \sum_{m=0}^{\infty} \frac{(iH)^m}{m!}$$
$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(iE_1)^m}{m!} & \cdots & \cdots & \cdots \\ 0 & \sum_{m=0}^{\infty} \frac{(iE_2)^m}{m!} & \cdots & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots & \cdots & \cdots & \sum_{m=0}^{\infty} \frac{(iE_n)^m}{m!} \end{pmatrix}$$
$$= \begin{pmatrix} e^{iE_1} & \cdots & \cdots & \cdots \\ \cdots & e^{iE_2} & \cdots & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots & \cdots & \cdots & e^{iE_n} \end{pmatrix} \tag{25}$$

Then, it's trivial to see that

$$Det(U) = e^{i(E1+E2+.....)} = e^{iTr(H)} \tag{26}$$

Problem 5

Let's simply do Taylor expansion.

$$Sin(\lambda\Omega) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(\lambda\Omega)^{2n-1}}{(2n-1)!} \tag{27}$$

$$Cos(\lambda\Omega) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(\lambda\Omega)^{2n-2}}{(2n-2)!} \tag{28}$$

Then, it is trivial to see that

$$\frac{dSin(\lambda\Omega)}{d\lambda} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\lambda^{2n-1}\Omega^{2n-1}}{(2n-2)!} = \Omega\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(\lambda\Omega)^{2n-2}}{(2n-2)!} = \Omega Cos(\lambda\Omega) \tag{29}$$