Physics 125a – Problem Set 4 – Due Oct 29, 2007
Solutions

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Problem 1

Using the fact that the delta function vanishes except near the intervals where the argument becomes zero, we can see that

$$\int_{-\infty}^{\infty} \delta\left(f(x)\right) g(x) dx = \sum_i \int_{x_i-\epsilon}^{x_i+\epsilon} \delta\left(f(x_i) + (x - x_i) f'(x_i) + \cdots\right) g(x) dx \tag{1}$$

The quantity ϵ does not need to go to zero in the end; it just needs to be small enough so that the integration intervals around the different x_i do not overlap. Since $f'(x_i) \neq 0$ for all x_i, the x_i are separated and we can always find such an ϵ.

Define $y_i = (x - x_i) f'(x_i)$, $dy_i = f'(x_i) dx$, and $\epsilon'_i = \epsilon\, f'(x_i)$ and do a change of variables for each integral in the sum:

$$\int_{-\infty}^{\infty} \delta\left(f(x)\right) g(x) dx = \sum_i \int_{-\epsilon'}^{\epsilon'} \delta(y_i) g(x(y_i)) \frac{dy_i}{f'(x_i)} \tag{2}$$

where $x(y_i)$ is obtained by inverting the definition of y_i. Now, if $f'(x_i) < 0$, the limits of integration will be in the wrong order and the integral picks up a negative sign, so let's sort that out:

$$\int_{-\infty}^{\infty} \delta\left(f(x)\right) g(x) dx = \sum_i \int_{-|\epsilon'|}^{|\epsilon'|} \delta(y_i) g(x(y_i)) \frac{dy_i}{|f'(x_i)|} \tag{3}$$

Now, do the integration:

$$\int_{-\infty}^{\infty} \delta\left(f(x)\right) g(x) dx = \sum_i \frac{g(x(y_i = 0))}{|f'(x_i)|} = \sum_i \frac{g(x_i)}{|f'(x_i)|} \tag{4}$$

The above result for the integration is the same as we would obtain if $\delta(f(x)) = \sum_i \delta(x - x_i) / |f'(x_i)|$; hence we have proven the formula.

Let's apply this result to prove the given equalities:

$$\delta(ax) = \frac{\delta(x-0)}{\left|\frac{d}{dx}(ax)\big|_{x=0}\right|} = \frac{1}{|a|}\delta(x) \tag{5}$$

$$\delta(x^2 - a^2) = \frac{\delta(x-a)}{\left|\frac{d}{dx}(x^2 - a^2)\big|_{x=a}\right|} + \frac{\delta(x+a)}{\left|\frac{d}{dx}(x^2 - a^2)\big|_{x=-a}\right|} = \frac{1}{2\,|a|}\left[\delta(x-a) + \delta(x+a)\right] \tag{6}$$

Problem 2

First, one must sort out what interval to integrate over. We have not defined the delta function for an imaginary argument – in fact, the representation in terms of an integral over complex exponentials fails miserably if we naively extend it to imaginary argument – so one should set the integration interval to be $[0, \infty]$. One might be worried that the place where the delta function might be singular is at the integration limit rather than contained in the interval. This is especially problematic if we use the two limiting forms given in the notes, Equations 3.135 and 3.136. But one must remember that the fundamental definition of the delta function is in terms of its action on a function it is integrated with, and that definition is consistent with the delta function's action being valid at the integration limit; *i.e.*,

$$\int_a^b dx\, \delta(x-a)\, f(x) = f(a) \tag{7}$$

For the first part of the problem, let's change the integration variable from x to $t = \sqrt{x}$,

$$\int_0^\infty \delta(\sqrt{x}) g(x) dx = \int_0^\infty \delta(t) g(t^2) 2t dt = \int_0^\infty \delta(t) h(t) dt \tag{8}$$

where we defined a new function $h(t) = 2t \cdot g(t)$. Because $h(t)$ vanishes at $t = 0$ and $\delta(t) = 0$ except at $t = 0$, the product of the two vanishes for the whole integration interval. This is the same result as when $\delta(\sqrt{x}) = 0$.

For the second part, we must again restrict the integration interval to ensure the argument of the delta function stays real. We then do a change of variables to $u = x^2 - a^2$,

$$\int_{-\infty}^{-a} \delta(\sqrt{x^2 - a^2}) g(x) dx + \int_a^\infty \delta(\sqrt{x^2 - a^2}) g(x) dx = 2\int_0^\infty \delta(\sqrt{u}) g(x(u)) \frac{1}{2x(u)} du = 0 \tag{9}$$

because $\delta(\sqrt{u}) = 0$ for the given integration interval and $x = 0$ is not inside the integration interval (so we don't have to worry that singularity).

Problem 3

$$\begin{aligned}
\int_{-\infty}^{\infty} \left[\frac{d}{dx}\theta(x - x')\right] f(x')dx' &= \int_{-\infty}^{\infty} \left[-\frac{d}{dx'}\theta(x - x')\right] f(x')dx' && (10)\\
&= -\int_{x-\epsilon}^{x+\epsilon} \left[\frac{d}{dx'}\theta(x - x')\right] f(x')dx' && (11)\\
&= -\theta(x - x')f(x')\Big|_{x'=x-\epsilon}^{x'=x+\epsilon} + \int_{x-\epsilon}^{x+\epsilon} \theta(x - x')\left[\frac{d}{dx'}f(x')\right] dx' && (12)\\
&= f(x - \epsilon) + \int_{x-\epsilon}^{x} \frac{d}{dx'} f(x')dx' && (13)\\
&= f(x - \epsilon) + [f(x) - f(x - \epsilon)] = f(x) && (14)
\end{aligned}$$

Therefore $\frac{d}{dx}\theta(x - x')$ has the same effect on any function under an integral sign that $\delta(x - x')$ does, so it holds that they are the same function.

Problem 4

Let's define $\delta_n(x-x')$ as

$$\delta_n(x-x') = \frac{n}{\sqrt{\pi}} e^{-n^2(x-x')^2} \tag{15}$$

Then $\delta(x-x') = \lim_{n\to\infty} \delta_n(x-x')$. By integrating by parts,

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{d}{dx}\delta(x-x')\right) f(x')dx' &= \lim_{n\to\infty}\left[\int_{-\infty}^{\infty}\left(\frac{d}{dx}\delta_n(x-x')\right) f(x')dx'\right] &(16)\\
&= \lim_{n\to\infty}\left[\int_{-\infty}^{\infty}\left(-\frac{d}{dx'}\delta_n(x-x')\right) f(x')dx'\right] &(17)\\
&= \lim_{n\to\infty}\left[-\delta_n(x-x')f(x')\Big|_{x'=-\infty}^{x'=\infty}\right.\\
&\qquad\left.+\int_{-\infty}^{\infty}\delta_n(x-x')\frac{d}{dx'}f(x')dx'\right] &(18)\\
&= \lim_{n\to\infty}\left[\int_{-\infty}^{\infty}\delta_n(x-x')\frac{d}{dx'}f(x')dx'\right] &(19)\\
&= \int_{-\infty}^{\infty}\delta(x-x')\frac{d}{dx'}f(x')dx' &(20)
\end{aligned}$$

Therefore

$$\frac{d}{dx}\delta(x-x') = \delta(x-x')\frac{d}{dx'} \tag{21}$$

Note that the proof is not really any different from what we did in class/lecture notes; the only distinction is that we did the integration by parts before taking the limit $n \to \infty$ so that we are sure our normal integration rules are valid.

Also, given

$$\lim_{n\to\infty}\left[\int_{-\infty}^{\infty}\left(-\frac{d}{dx'}\delta_n(x-x')\right) f(x')dx'\right] = -\int_{-\infty}^{\infty}\left(\frac{d}{dx'}\delta(x-x')\right) f(x')dx' \tag{22}$$

and Equation (17), we can see that

$$\frac{d}{dx'}\delta(x-x') = -\delta(x-x')\frac{d}{dx'} \tag{23}$$

Finally, we admit this was not the best problem in the world – the limit definition of the delta function is not really valid, so claiming that the above proof is more rigorous than what we did in class is not really correct.

Problem 5

From the completeness and the orthonomality of the $\{|k\rangle\}$ basis, we know that

$$I = \int_{-\infty}^{\infty} dk |k\rangle\langle k| \tag{24}$$

Therefore, by inserting the identity operator between the bra and the ket, we can show that

$$\langle f|g\rangle = \langle f|I|g\rangle = \int_{-\infty}^{\infty} dk \langle f|k\rangle\langle k|g\rangle = \int_{-\infty}^{\infty} dk \widetilde{f}^*(k)\widetilde{g}(k) \tag{25}$$

Or, if we want to start from the $\{|x\rangle\}$ basis,

$$\begin{aligned}
\langle f|g\rangle &= \int_{-\infty}^{\infty} dx \langle f|x\rangle\langle x|g\rangle &(26)\\
&= \int_{-\infty}^{\infty} dx \left(\int_{-\infty}^{\infty} dk \langle f|k\rangle\langle k|x\rangle\right)\left(\int_{-\infty}^{\infty} dk' \langle x|k'\rangle\langle k'|g\rangle\right) &(27)\\
&= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dx \left[\frac{1}{2\pi} e^{-i(k-k')x}\right] \widetilde{f}^*(k)\widetilde{g}(k') &(28)\\
&= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \delta(k-k')\widetilde{f}^*(k)\widetilde{g}(k') &(29)\\
&= \int_{-\infty}^{\infty} dk \widetilde{f}^*(k)\widetilde{g}(k) &(30)
\end{aligned}$$