These problems cover Shankar 7, 9, and 16 and Lecture Notes 6, 7, and 8 on the simple harmonic oscillator, uncertainty relations, and approximation methods.

Because you have 1.5 weeks for it, this set is longer than usual. Please plan accordingly!

**Version 2:**

Problem 4: There was a typo in the energy for $V_0$. Also, obtain the energies numerically, not in closed form analytically.

Problem 6: We covered the variational technique, so you do have to do this problem. There were some problems with the original version of this one (wrong trial wavefunction, cubic equation for $r_0$, etc.), which have now been corrected.

1. Shankar 7.4.2 and 7.4.3. Note the very simple structure of $X$ and $P$ in the energy basis (Shankar Equations 7.4.32 and 7.4.33).

2. Shankar 7.5.4. Be careful to avoid confusing Boltzmann’s constant $k$ with the spring constant of the oscillator (which, in this problem is given as $m\omega^2$).

3. Consider an attractive $\delta$-function potential, $V(x) = -aV_0\delta(x)$, $V_0 > 0$. Such a system always has a bound state of energy $E = -ma^2V_0^2/2\hbar^2$. Use the uncertainty principle to obtain this energy. Hints:
   - You should assume that the wavefunction is exponential everywhere (based on the fact $E < V(x)$ for $x \neq 0$), and use this fact to help you evaluate $(\Delta X)^2$.
   - At some point, you will need to calculate $\langle V \rangle = \langle -V_0a\delta(x) \rangle$. Just assuming an exponential wavefunction provides you enough information to evaluate this; you can even evaluate it using only dimensional arguments.
   - Look back at how Shankar found the energy of the hydrogen atom ground state in Shankar Section 9.4.
4. Consider the double-well potential shown in the figure.

\[ V(z) = \lim_{V_i \to \infty} \begin{cases} 
V_0 & |z| < \Delta L/2 \\
0 & \Delta L/2 \leq |z| \leq L_z/2 \\
V_1 & |z| > L_z/2 
\end{cases} \]

\[ L_z = 3 \text{ nm} \]
\[ \Delta L = 0.05 \text{ nm} \]
\[ V_0 = 1 \text{ meV} \]

Use the WKB approximation to obtain equations that determine the lowest two energy states of the system. You don’t need to solve these analytically, but you should calculate the numerical results for the lowest two energy states and compare them to the \(V_0 = 0\) case. (Note that, in spite of the sharp step, the WKB approximation remains valid because \(\Delta V = V_0\) is small compared to the kinetic energy.) Then, suppose a constant electric field \(F\) is applied, keeping the potential energy at the center of the well fixed but inducing a tilt in the well. You should make the approximation that the energy shift due to the applied field is small. Show that the energy of the ground state changes quadratically with applied field; that is,

\[ \Delta E_0 = -\frac{1}{2} \alpha F^2 \]

\(\alpha\) is the polarizability. You do not need to obtain a closed form for \(\alpha\); you only need to show that the leading order correction to the energy has a quadratic dependence on \(F\).

A physical example of this system is the two-level defect system we discussed in Problem Set 3 Problem 1. The polarizability of this system is how an interaction with the electromagnetic field arises.
5. In the phenomenon of cold emission, electrons are drawn from a metal (at room temperature) by an externally supported electric field. The potential well that the metal presents to the free electrons before the electric field is turned on is shown in the left figure below; it is a square well of depth $E_F + \Phi$, but the states up to $E_F$ are filled by electrons ($E_F$ is called the Fermi level), so the most energetic electrons are an energy $\Phi$ (the work function) away from being free. After application of the electric field $\mathcal{E}$, the potential at the surface slopes down as shown in the right figure below, thereby electrons to “tunnel” through the potential barrier. If the surface of the metal is taken as the $x = 0$ plane, the new potential outside the surface is

$$V(x) = \Phi + E_F - e\mathcal{E}x$$

(a) Use the WKB approximation to calculate the transmission coefficient for cold emission.

(b) Estimate the field strength, $\mathcal{E}$, in V/cm, necessary to draw a current density of the order of mA/cm$^2$ from a potassium surface. Do this by assuming that the current density incident on the barrier from the left is $J = e n v$ where $v$ is the velocity corresponding to $E_F$ ($E_F = \frac{1}{2} m v^2$) and $n$ is the electron density. For potassium, $E_F = 2.1$ eV, $\Phi = 2.1$ eV (that’s not a typo, they are indeed equal, but that’s just a coincidence), and the Fermi energy and electron density are related by

$$E_F = \frac{\hbar^2}{2 m} \left( \frac{3 n}{8 \pi} \right)^{2/3}$$
6. A Yukawa potential is a potential of the form

\[ V(x) = -\alpha e^{-\kappa r} \quad \alpha, \kappa > 0 \quad (4) \]

For small radii, it looks like a Coulomb potential, but for large radii it decays away. This potential is the one that describes the potential for a force mediated by a massive particle (e.g., the weak force): the decay constant \( \kappa \) is proportional to the particle mass. For electromagnetism, which is mediated by the massless photon, \( \kappa = 0 \) and one recovers the Coulomb potential.

We will show later that, for spherically symmetric problems in three dimensions, the position-basis eigenstates have the form

\[ \psi_{E,l,m}(r, \theta, \phi) = R_{E,l}(r)Y_{l,m}(\theta, \phi) \quad (5) \]

where \( R_{E,l}(r) \) is the radial wavefunction and that the eigenvector-eigenvalue equation for the Hamiltonian becomes the differential equation

\[ -\frac{\hbar^2}{2\mu} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} R_{E,l}(r) \right) \right] - \frac{l(l+1)}{r^2} R_{E,l}(r) + V(r)R_{E,l}(r) = ER_{E,l}(r) \quad (6) \]

where \( \mu \) is the particle mass and \( l \) describes the state’s total angular momentum – such states are eigenstates of the total angular momentum squared operator \( L^2 \) with eigenvalues \( \hbar^2 l(l+1) \).

Use the variational technique to estimate the ground state energy for the Yukawa potential for \( l = 0 \) and for \( \kappa \) small (i.e., close to a Coulomb potential). You should use an exponentially decaying trial wavefunction of the form

\[ R(r) \propto e^{-r/r_0} \quad (7) \]

and note that, because we are working in three dimensions, the expectation value for the Hamiltonian in a \( l = 0 \) state is

\[ \langle \psi | H | \psi \rangle = \int_0^\infty r^2 dr R^*(r) \left\{ -\frac{\hbar^2}{2\mu} \left[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} R(r) \right) \right] + V(r)R(r) \right\} \quad (8) \]

and the normalization of the wavefunction is

\[ \langle \psi | \psi \rangle = \int_0^\infty r^2 dr R^*(r)R(r) \quad (9) \]

You will find it useful to write your solution as a correction to the standard hydrogen atom solution (\( \kappa = 0 \) exactly), for which

\[ r_0 = a_0 = \frac{\hbar^2}{\mu \alpha} \quad E_0 = -\frac{\mu \alpha^2}{2\hbar^2} = -\frac{\alpha}{2r_0} \quad (10) \]

Since \( \kappa \) has dimensions of 1/length, obtain a dimensionless criterion for what is meant by \( \kappa \) being small. **Hints:** 1) Before doing any Taylor expansions, rewrite your expressions in terms of a dimensionless combination of \( \kappa \) and some other physical parameter so that it is clear which terms are small and which are not; 2) Be careful to carry your Taylor expansions to high enough order; if you do an expansion to insufficiently high order, you will just get \( a_0 \) and \( E_0 \) for your results.