Physics 125a – Problem Set 7 – Due Nov 27, 2007

Version 2 – Nov 20, 2007

Note the delayed due date!

This problem set focuses on uncertainty relations and on multiparticle systems (Shankar 9 and 10.1-10.2, Lecture Notes 7 and 8.1-8.3).

Many basic problems in QM can be found in textbooks – there are only so many solvable elementary problems out there. Please refrain from using solutions from other textbooks. Obviously, you will learn more and develop better intuition for QM by solving the problems yourself. We are happy to provide hints to get you through the tricky parts of a problem, but you *must* learn to set up and solve these problems from scratch by yourself.

v. 2: A bit of clarification on how stringent your uncertainty bound must be in Problem 1. Rewrite Problem 5 as a counterexample since it turns out to be false.

1. For a single particle subject to the generic one-dimensional Hamiltonian

$$H = \frac{P^2}{2\,m} + V(X) \tag{1}$$

   obtain uncertainty relations for the following products:

   (a) $\sqrt{(\Delta X)^2\,(\Delta E)^2}$

   (b) $\sqrt{(\Delta P)^2\,(\Delta E)^2}$

   (c) $\sqrt{(\Delta X)^2\,(\Delta T)^2}$

   (d) $\sqrt{(\Delta P)^2\,(\Delta T)^2}$

   where the particle's kinetic energy expectation value is T and total energy expectation value is E. Don't try to evaluate the anticommutator term given in the generic uncertainty relation (Equation 7.6 of the lecture notes); use the less stringent bound given by neglecting the anticommutator term, like what we did to obtain Equation 7.8 of the lecture notes. (Hint: what operators' expectation values are being taken to give T and E?)

2. Ignore the fact that the hydrogen atom is a three-dimensional system and pretend that

$$H = \frac{P^2}{2\,m} - \frac{e^2}{(R^2)^{1/2}} \qquad \text{with} \quad P^2 = P_X^2 + P_Y^2 + P_Z^2, \quad R^2 = X^2 + Y^2 + Z^2 \tag{2}$$

   corresponds to a one-dimensional problem. Assuming

$$\sqrt{(\Delta P)^2\,(\Delta R)^2} \geq \frac{\hbar}{2} \tag{3}$$

   estimate the ground-state energy.

3. The Baker-Hausdorf Lemma. Show that, for operators A and B, with $[A,[A,B]] = 0$ and $[B,[A,B]] = 0$,

$$e^A\, e^B = e^{A+B}\, e^{\frac{1}{2}[A,B]} \tag{4}$$

Hint: First show that $[e^{\eta A}, B] = \eta\, e^{\eta A}[A,B]$ where η is a number, not an operator. Then estalblish that the derivative of

$$g(\eta) = e^{\eta A}\, e^{\eta B}\, e^{-\eta(A+B)} \tag{5}$$

is

$$\frac{dg}{d\eta} = \eta\,[A,B]\, g(\eta) \tag{6}$$

and integrate.

This relation is used in quantum statistical mechanics in the form

$$e^{\beta(A+B)} \approx e^{\beta A}\, e^{\beta B}\, e^{-\frac{1}{2}\beta^2\,[A,B]} \tag{7}$$

for $\beta \ll 1$, where β plays the role of inverse temperature and $A+B$ is the Hamiltonian.

4. Quantize the two-dimensional simple harmonic oscillator, for which the classical Hamiltonian is

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2\,m} + \frac{1}{2}\,m\,\omega_x^2 x^2 + \frac{1}{2}\,m\,\omega_y^2 y^2 \tag{8}$$

   (a) Show that the allowed energies are

$$E = \left(n_x + \frac{1}{2}\right)\hbar\,\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\,\omega_y \qquad n_x,\ n_y = 0,\ 1,\ 2,\ldots \tag{9}$$

   (b) Write down the corresponding position-basis representations (wavefunctions) in terms of single oscillator wavefunctions. Verify that they are eigenstates of the two-dimensional parity operator, Π, which performs the transformation $(x \to -x, y \to -y)$, and that the eigenvalues depend only on $n_x + n_y$.

   (c) Consider next the isotropic oscillator ($\omega_x = \omega_y$). Write *explicit*, normalized eigenfunctions of the first three states (that is, for the cases $n = 0$ and $n = 1$). Reexpress your results in terms of polar coordinates ρ and ϕ. For arbitary n, show that the degeneracy of a level with $E = (n+1)\,\hbar\,\omega$ is $n+1$.

5. Why you should always treat textbooks with skepticism: The following is Liboff Problem 8.32. It turns out that it is incorrect even though it seems like an entirely plausible result. Find a counterexample. Why does this not contradict the discussion in the subsection **Determination of Particle Statistics** in Section 10.3 of Shankar?

   Consider two identical particles in a one-dimensional box extending from $-L/2$ to $L/2$. Consider some distinguishable particle state $|\psi\,\rangle = |\psi_a\,\rangle^{(1)} \otimes |\psi_b\,\rangle^{(2)}$ where $|\psi_a\,\rangle$ and $|\psi_b\,\rangle$ are single-particle eigenstates with mode indices a and b. Let $|\psi\,\rangle_+$ and $|\psi\,\rangle_-$ denote the symmetrized and antisymmetrized version of this state. Calculate the expectation value for the square of the interparticle displacement $\left(2\,\delta X^{(1)\otimes(2)}\right)^2$ for the two states $|\psi\rangle_+$ and $|\psi\rangle_-$, where $\delta X^{(1)\otimes(2)}$ was defined in the lecture notes and you should write your expressions in terms of the single-particle matrix elements such as

$$X_{mn} = \langle\psi_m|X|\psi_n\rangle \qquad Y_{mn} = \langle\psi_m|X^2|\psi_n\rangle \tag{10}$$

where m and n may take on the values a or b (*i.e.*, $mn = aa$, ab, ba, and bb are allowed). (Note that $Y_{mn} \neq (X_{mn})^2$!) Show that

$$_+\langle\psi|\left(\delta X^{(1)\otimes(2)}\right)^2|\psi\rangle_+ \leq \ _-\langle\psi|\left(\delta X^{(1)\otimes(2)}\right)^2|\psi\rangle_- \tag{11}$$

thus establishing that (in a statistical sense) particles in a symmetric state attract one another while particles in an antisymmetric state repel one another. Such attractions and repulsions are termed *exchange phenomena*.