Physics 125a – Problem Set 7 – Due Nov 27, 2007
Solutions

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v. 2: corrections to eqns 66, 67, 68.

Problem 1

When $[A,[A,B]] = [B,[A,B]] = 0$,

$$\begin{aligned}
[A,B^n] &= [A,\underbrace{BBB\cdots B}_{n}] && (1)\\
&= ABBB\cdots B - BBB\cdots BA && (2)\\
&= ABBB\cdots B + (-BABB\cdots B + BABB\cdots B) + (-BBAB\cdots B + BBAB\cdots B) && (3)\\
&\quad + \cdots - BBB\cdots A && (4)\\
&= [A,B]\,B\cdots B + B\,[A,B]\,B\cdots B + \cdots + B\cdots B\,[A,B] && (5)\\
&= B\cdots B\,[A,B] + B\cdots B\,[A,B] + \cdots + B\cdots B\,[A,B] \quad (\because [B,[A,B]] = 0) && (6)\\
&= n{\cdot}B^{n-1}\,[A,B] && (7)
\end{aligned}$$

Then, for $f(B) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)} B^n$, where $f^{(n)}$ are some numbers,

$$\begin{aligned}
[A,f(B)] &= \left[A, \left\{\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)} B^n\right\}\right] && (8)\\
&= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}\,[A,B^n] && (9)\\
&= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}{\cdot}nB^{n-1}\,[A,B] && (10)\\
&= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} f^{(n)} B^{n-1}\,[A,B] = \frac{df(B)}{dB}\,[A,B] && (11)
\end{aligned}$$

where by $\frac{df(B)}{dB}$ we mean treat B as a number for the purpose of taking the derivative. In (11) the order of derivative and $[A,B]$ is not important because $[B,[A,B]] = 0$.

(a)

$$\begin{aligned}\sqrt{(\Delta X)^2(\Delta E)^2} &\geq \frac{1}{2}\left|\langle[X,H]\rangle\right| && (12)\\ &= \frac{1}{2}\left|\left\langle\left[X,\frac{P^2}{2m}+V(X)\right]\right\rangle\right| && (13)\\ &= \frac{1}{2}\left|\left\langle\left[X,\frac{P^2}{2m}\right]\right\rangle\right| \quad (\because X \text{ commutes with } V(X)) && (14)\\ &= \frac{1}{2}\left|\left\langle\frac{P}{m}[X,P]\right\rangle\right| && (15)\\ &= \frac{\hbar}{2m}\left|\langle P\rangle\right| && (16)\end{aligned}$$

We used the lemma proved at the top to go from (15) to (16).

(b)

$$\begin{aligned}\sqrt{(\Delta P)^2(\Delta E)^2} &\geq \frac{1}{2}\left|\langle[P,H]\rangle\right| && (17)\\ &= \frac{1}{2}\left|\left\langle\left[P,\frac{P^2}{2m}+V(X)\right]\right\rangle\right| && (18)\\ &= \frac{1}{2}\left|\langle[P,V(X)]\rangle\right| && (19)\\ &= \frac{1}{2}\left|\frac{dV(X)}{dX}\langle[P,X]\rangle\right| && (20)\\ &= \frac{\hbar}{2}\left|\left\langle\frac{dV(X)}{dX}\right\rangle\right| && (21)\\ &= \frac{\hbar}{2}\left|\left\langle\sum_{n=0}^{\infty}\frac{1}{n!}V^{(n+1)}X^n\right\rangle\right| && (22)\end{aligned}$$

where $V(X)=\sum_{n=0}^{\infty}\frac{1}{n!}V^{(n)}X^n$ for some numbers $V^{(n)}$.

Also, we can calculate $\langle[P,V(X)]\rangle$ using the position basis representation of $\langle x|P|x'\rangle$, which is $-i\hbar\frac{d}{dx}\delta(x-x')$.

$$\begin{aligned}\langle[P,V(X)]\rangle &= \langle P\cdot V(X)\rangle-\langle V(X)\cdot P\rangle && (23)\\ &= \int_{-\infty}^{\infty}dx\psi^*(x)\left[-i\hbar\frac{d}{dx}\cdot V(x)-V(x)\left(-i\hbar\frac{d}{dx}\right)\right]\psi(x) && (24)\\ &= \int_{-\infty}^{\infty}dx\psi^*(x)\left(-i\hbar\frac{dV(x)}{dx}\right)\psi(x) && (25)\\ &= \left\langle -i\hbar\frac{dV(x)}{dx}\right\rangle && (26)\end{aligned}$$

(c)

$$\begin{aligned}\sqrt{(\Delta X)^2(\Delta T)^2} &\geq \frac{1}{2}\left|\langle[X,T]\rangle\right| && (27)\\ &= \frac{1}{2}\left|\left\langle\left[X,\frac{P^2}{2m}\right]\right\rangle\right| && (28)\\ &= \frac{\hbar}{2m}\left|\langle P\rangle\right| && (29)\end{aligned}$$

(d)

$$\begin{aligned}\sqrt{(\Delta P)^2(\Delta T)^2} &\geq \frac{1}{2}\left|\langle[P,T]\rangle\right| && (30)\\ &= \frac{1}{2}\left|\left\langle\left[P,\frac{P^2}{2m}\right]\right\rangle\right| && (31)\\ &= 0 && (32)\end{aligned}$$

Problem 2

By symmetry,

$$\langle P\rangle = \langle R\rangle = 0 \tag{33}$$

Let's assume that the ground state is a state with the minimum uncertainty, $\hbar/2$. Then,

$$(\Delta P)^2(\Delta R)^2 = \langle P^2\rangle\langle R^2\rangle = \frac{\hbar^2}{4} \tag{34}$$

Therefore,

$$\langle H\rangle = \frac{1}{2m}\langle P^2\rangle - \frac{e^2}{\sqrt{\langle R^2\rangle}} = \frac{1}{2m}\langle P^2\rangle - \frac{2e^2}{\hbar}\sqrt{\langle P^2\rangle} \tag{35}$$

This has a minimum at $\langle P^2\rangle = 2me^2/\hbar$, giving $\langle H\rangle = -2me^4/\hbar^2$. Thus the ground-state energy will be of the same order of magnitude as this. This estimate is 4 times larger in magnitude than true value due to treating P and R as one-dimensional operators – the estimate in Shankar was closer to the actual value because uncertainties were treated more carefully.

Problem 3

Let's use the operator identity proven in Problem 1, $[A, f(B)] = \frac{df(B)}{dB}[A,B]$. Then,

$$\begin{aligned}\left[e^{\eta A}, B\right] &= \eta e^{\eta A}[A,B] && (36)\\ \left[A, e^{\eta B}\right] &= \eta e^{\eta B}[A,B] && (37)\end{aligned}$$

Even if you did not manage to discover the derivative part of the identity, you will have had to taylor expand $e^{\eta A}$, figure out $[A, B^n]$, and then an obvious resumming of the series would have given you these results.

$$\begin{aligned}\frac{dg(\eta)}{d\eta} &= Ae^{\eta A}\cdot e^{\eta B}\cdot e^{-\eta(A+B)} + e^{\eta A}\cdot Be^{\eta B}\cdot e^{-\eta(A+B)} + e^{\eta A}\cdot e^{\eta B}\cdot(-A-B)\,e^{-\eta(A+B)} && (38)\\ &= Ae^{\eta A}e^{\eta B}e^{-\eta(A+B)} + e^{\eta A}Be^{\eta B}e^{-\eta(A+B)} - e^{\eta A}e^{\eta B}Ae^{-\eta(A+B)} - e^{\eta A}e^{\eta B}Be^{-\eta(A+B)} && (39)\end{aligned}$$

Commute B and $e^{\eta B}$, note that terms cancel:

$$\begin{aligned}&= Ae^{\eta A}e^{\eta B}e^{-\eta(A+B)} - e^{\eta A}\cdot e^{\eta B}A\cdot e^{-\eta(A+B)} && (40)\\ &= Ae^{\eta A}e^{\eta B}e^{-\eta(A+B)} - e^{\eta A}\left(Ae^{\eta B} - \eta e^{\eta B}[A,B]\right)e^{-\eta(A+B)} && (41)\\ &= \eta[A,B]\,e^{\eta A}e^{\eta B}e^{-\eta(A+B)} \quad (\because [A,[A,B]] = [B,[A,B]] = 0) && (42)\\ &= \eta[A,B]\,g(\eta) && (43)\end{aligned}$$

For (38), note that there is degeneracy in the ordering because $[C, e^{\eta}C] = 0$ for any C, but that any method of doing the ordering will give the same result.

One might be tempted to apply $[g(\eta)]^{-1}$ to both sides, but we have the problem that we don't know how to integrate $[g(\eta)]^{-1}\frac{dg}{d\eta}$. You of course would guess that this yields $\log(g(\eta))$, but we have never shown that. Rather than proving that generically, let's just use Taylor series expansion, our standard technique for evaluating functions of operators, to relate the two sides.

For $g(\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)} \eta^n$, where $g^{(0)} = I$ and $g^{(n)}$ are some functions of A and B,

$$\begin{aligned} \frac{dg(\eta)}{d\eta} = \frac{d}{d\eta}\left[\sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)} \eta^n\right] &= g^{(1)} + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} g^{(n+2)} \eta^{n+1} && (44) \\ \eta\,[A,B]\, g(\eta) = \eta\,[A,B]\left[\sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)} \eta^n\right] &= \left[\sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)} \eta^{n+1}\right] && (45) \end{aligned}$$

and after term-by-term comparison, we can see that

$$\begin{aligned} g^{(1)} &= 0 && (46) \\ \frac{1}{(n+1)!} g^{(n+2)} &= \frac{1}{n!}\,[A,B]\, g^{(n)} && (47) \end{aligned}$$

Using this recurrence relation with $g^{(0)} = I$ and $g^{(1)} = 0$, we can find that

$$g^{(n)} = (n-1)\,[A,B]\, g^{(n-2)} = \cdots = \begin{cases} (n-1)!!\,([A,B])^{n/2}\, g^{(0)} & \text{for even} \quad n \\ (n-1)!!\,([A,B])^{(n-1)/2}\, g^{(1)} = 0 & \text{for odd} \quad n \end{cases} \qquad (48)$$

where $n!! = n\cdot(n-2)\cdot(n-4)\cdots$.

Therefore,

$$\begin{aligned} g(\eta) &= \sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)} \eta^n && (49) \\ &= \sum_{2m=0}^{\infty} \frac{1}{(2m)!}\,(2m-1)!!\,([A,B])^{2m/2}\, g^{(0)} \eta^{2m} && (50) \\ &= \sum_{m=0}^{\infty} \frac{1}{2^m m!}\,([A,B])^m\, g^{(0)}\,(\eta^2)^m && (51) \\ &= \sum_{m=0}^{\infty} \frac{1}{m!}\left([A,B]\,\frac{\eta^2}{2}\right)^m && (52) \\ &= e^{[A,B]\frac{\eta^2}{2}} && (53) \end{aligned}$$

You could have proven this also by guessing the result based on your expectation for how to integrate $[g(\eta)]^{-1}\frac{dg}{d\eta}$, and then checking it via Taylor expansion.

Equating this with $g(\eta) = e^{\eta A} e^{\eta B} e^{-\eta(A+B)}$,

$$e^{\eta A} e^{\eta B} = e^{[A,B]\frac{\eta^2}{2}} e^{\eta(A+B)} = e^{\eta(A+B)} e^{[A,B]\frac{\eta^2}{2}} \qquad (54)$$

where the last equality holds because $[A,[A,B]] = [B,[A,B]] = 0$. Now if we substitute $\eta = 1$ into the result, we can prove the lemma.

Problem 4

(a) As H is separable into two independent $H_x = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_x^2 x^2$ and $H_y = \frac{p_y^2}{2m} + \frac{1}{2}m\omega_y^2 y^2$,

$$E = E_x + E_y = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y \tag{55}$$

$$\psi_{(n_x,n_y)} = \psi_{n_x}(x) \cdot \psi_{n_y}(y) \tag{56}$$

where E_i and ψ_i are the usual energy eigenvalue and eigenstate of a simple harmonic oscillator with $\omega = \omega_i$.

(b) We have seen from (a) that $\psi_{(n_x,n_y)} = \psi_{n_x}(x) \cdot \psi_{n_y}(y)$. And we know that

$$\Pi_x \psi_{n_x}(x) = (-1)^{n_x} \psi_{n_x}(x) \tag{57}$$

$$\Pi_y \psi_{n_y}(y) = (-1)^{n_y} \psi_{n_y}(y) \tag{58}$$

Combining these two facts, we can see that

$$\begin{aligned}\Pi\psi_{(n_x,n_y)} &= (\Pi_x \otimes \Pi_y)\, \psi_{n_x}(x)\, \psi_{n_y}(y) & (59)\\ &= \Pi_x \psi_{n_x}(x) \cdot \Pi_y \psi_{n_y}(y) & (60)\\ &= (-1)^{n_x} \psi_{n_x}(x) \cdot (-1)^{n_y} \psi_{n_y}(y) & (61)\\ &= (-1)^{n_x+n_y} \psi_{(n_x,n_y)} & (62)\end{aligned}$$

(c) The wavefunctions of a simple harmonic oscillator with ω are, for $\alpha = \sqrt{m\omega/\hbar}$,

$$\psi_n(x) = \left(\frac{\alpha^2}{2^{2n}(n!)^2\pi}\right)^{1/4} H_n(\alpha x)\, e^{-\alpha^2 x^2/2} \tag{63}$$

$$H_0(\alpha x) = 1 \tag{64}$$

$$H_1(\alpha x) = 2\alpha x \tag{65}$$

When $\omega_x = \omega_y = \omega$, the wavefunctions $\psi_{(n_x,n_y)}$ of the first three states are

$$\psi_{(0,0)} = \alpha\sqrt{\frac{1}{\pi}} e^{-\alpha^2(x+y)^2/2} = \alpha\sqrt{\frac{1}{\pi}} e^{-\alpha^2\rho^2/2} \tag{66}$$

$$\psi_{(1,0)} = \alpha^2\sqrt{\frac{2}{\pi}} x e^{-\alpha^2(x^2+y^2)/2} = \alpha^2\sqrt{\frac{2}{\pi}} \rho\cos(\phi)\, e^{-\alpha^2\rho^2/2} \tag{67}$$

$$\psi_{(0,1)} = \alpha^2\sqrt{\frac{2}{\pi}} y e^{-\alpha^2(x^2+y^2)/2} = \alpha^2\sqrt{\frac{2}{\pi}} \rho\sin(\phi)\, e^{-\alpha^2\rho^2/2} \tag{68}$$

For this isotropic oscillator, $E = E_x + E_y = (n_x + n_y + 1)\hbar\omega$. Thus for the same $n_x + n_y = n$, the energy eigenvalue is the same. As there are $n+1$ ways to split n into two nonnegative integers, the degeneracy is $n+1$.

Problem 5

$$\begin{aligned}
\left(2\delta X^{(1)\otimes(2)}\right)^2 &= \left(X^{(1)}\otimes I^{(2)} - I^{(1)}\otimes X^{(2)}\right)^2 \\
&= \left(X^{(1)}\otimes I^{(2)}\right)^2 - \left(X^{(1)}\otimes I^{(2)}\right)\left(I^{(1)}\otimes X^{(2)}\right) && (69)\\
&\quad - \left(I^{(1)}\otimes X^{(2)}\right)\left(X^{(1)}\otimes I^{(2)}\right) + \left(I^{(1)}\otimes X^{(2)}\right)^2 && (70)\\
&= \left(X^{(1)}\right)^2 \otimes \left(I^{(2)}\right)^2 - \left(X^{(1)}I^{(1)}\right)\otimes\left(I^{(2)}X^{(2)}\right) \\
&\quad - \left(I^{(1)}X^{(1)}\right)\otimes\left(X^{(2)}I^{(2)}\right) + \left(I^{(1)}\right)^2\otimes\left(X^{(2)}\right)^2 && (71)\\
&= \left(X^{(1)}\right)^2\otimes I^{(2)} - 2X^{(1)}\otimes X^{(2)} + I^{(1)}\otimes\left(X^{(2)}\right)^2 && (72)\\
&= Y^{(1)}\otimes I^{(2)} - 2X^{(1)}\otimes X^{(2)} + I^{(1)}\otimes Y^{(2)} && (73)
\end{aligned}$$

For $|\psi\rangle = \frac{1}{2}\left(|\psi_a\rangle\otimes|\psi_b\rangle + \gamma|\psi_b\rangle\otimes|\psi_a\rangle\right)$, $|\psi\rangle = |\psi_+\rangle$ when $\gamma = 1$, and $|\psi\rangle = |\psi_-\rangle$ when $\gamma = -1$. Then, as $I_{ab} = I_{ba} = 0$ and $X_{ab} = X_{ba}$,

$$\begin{aligned}
&\langle\psi|\left(2\delta X^{(1)\otimes(2)}\right)^2|\psi\rangle && (74)\\
&= \langle\psi|\left(Y^{(1)}\otimes I^{(2)}\right)|\psi\rangle - 2\langle\psi|\left(X^{(1)}\otimes X^{(2)}\right)|\psi\rangle + \langle\psi|\left(I^{(1)}\otimes Y^{(2)}\right)|\psi\rangle && (75)\\
&= \frac{1}{4}\left(\langle\psi_a|\otimes\langle\psi_b| + \gamma\langle\psi_b|\otimes\langle\psi_a|\right)\left(Y^{(1)}\otimes I^{(2)}\right)\left(|\psi_a\rangle\otimes|\psi_b\rangle + \gamma|\psi_b\rangle\otimes|\psi_a\rangle\right)\\
&\quad - 2\cdot\frac{1}{4}\left(\langle\psi_a|\otimes\langle\psi_b| + \gamma\langle\psi_b|\otimes\langle\psi_a|\right)\left(X^{(1)}\otimes X^{(2)}\right)\left(|\psi_a\rangle\otimes|\psi_b\rangle + \gamma|\psi_b\rangle\otimes|\psi_a\rangle\right)\\
&\quad + \frac{1}{4}\left(\langle\psi_a|\otimes\langle\psi_b| + \gamma\langle\psi_b|\otimes\langle\psi_a|\right)\left(I^{(1)}\otimes Y^{(2)}\right)\left(|\psi_a\rangle\otimes|\psi_b\rangle + \gamma|\psi_b\rangle\otimes|\psi_a\rangle\right) && (76)\\
&= \frac{1}{4}\Big(\langle\psi_a|Y^{(1)}|\psi_a\rangle\langle\psi_b|I^{(2)}|\psi_b\rangle + \gamma\langle\psi_a|Y^{(1)}|\psi_b\rangle\langle\psi_a|I^{(2)}|\psi_b\rangle\\
&\quad +\gamma\langle\psi_b|Y^{(1)}|\psi_a\rangle\langle\psi_b|I^{(2)}|\psi_a\rangle + \gamma^2\langle\psi_b|Y^{(1)}|\psi_b\rangle\langle\psi_b|I^{(2)}|\psi_b\rangle\Big)\\
&\quad - 2\cdot\frac{1}{4}\Big(\langle\psi_a|X^{(1)}|\psi_a\rangle\langle\psi_b|X^{(2)}|\psi_b\rangle + \gamma\langle\psi_a|X^{(1)}|\psi_b\rangle\langle\psi_a|X^{(2)}|\psi_b\rangle\\
&\quad +\gamma\langle\psi_b|X^{(1)}|\psi_a\rangle\langle\psi_b|X^{(2)}|\psi_a\rangle + \gamma^2\langle\psi_b|X^{(1)}|\psi_b\rangle\langle\psi_b|X^{(2)}|\psi_b\rangle\Big)\\
&\quad + \frac{1}{4}\Big(\langle\psi_a|I^{(1)}|\psi_a\rangle\langle\psi_b|Y^{(2)}|\psi_b\rangle + \gamma\langle\psi_a|I^{(1)}|\psi_b\rangle\langle\psi_a|Y^{(2)}|\psi_b\rangle\\
&\quad +\gamma\langle\psi_b|I^{(1)}|\psi_a\rangle\langle\psi_b|Y^{(2)}|\psi_a\rangle + \gamma^2\langle\psi_b|I^{(1)}|\psi_b\rangle\langle\psi_b|Y^{(2)}|\psi_b\rangle\Big) && (77)\\
&= \frac{1}{4}\left(Y_{aa}I_{bb} + \gamma Y_{ab}I_{ab} + \gamma Y_{ba}I_{ba} + \gamma^2 Y_{bb}I_{bb}\right) - 2\cdot\frac{1}{4}\left(X_{aa}X_{bb} + \gamma X_{ab}X_{ab} + \gamma X_{ba}X_{ba} + \gamma^2 X_{bb}X_{bb}\right)\\
&\quad + \frac{1}{4}\left(I_{aa}Y_{bb} + \gamma I_{ab}Y_{ab} + \gamma I_{ba}Y_{ba} + \gamma^2 I_{bb}Y_{bb}\right) && (78)\\
&= \frac{1}{4}\left(Y_{aa} + \gamma^2 Y_{bb}\right) - 2\cdot\frac{1}{4}\left(2\gamma X_{ab}^2 + \left(1+\gamma^2\right)X_{aa}X_{bb}\right) + \frac{1}{4}\left(Y_{bb} + \gamma^2 Y_{aa}\right) && (79)\\
&= \frac{1}{4}\left(1+\gamma^2\right)\left(Y_{aa} + Y_{bb} - 2X_{aa}X_{bb}\right) - \gamma X_{ab}^2 && (80)
\end{aligned}$$

Therefore $\langle\psi_+|\left(2\delta X^{(1)\otimes(2)}\right)^2|\psi_+\rangle \le \langle\psi_-|\left(2\delta X^{(1)\otimes(2)}\right)^2|\psi_-\rangle$.