Physics 125a – Problem Set 8 – Due Dec 7, 2007
Solutions

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December 8, 2007

Problem 1

Since we have two identical bosons, the state vector represents two particle state must be symmetric under the exchange of two particles. So, we have

$$|\psi\phi\rangle = \frac{1}{N}(|\phi\rangle \otimes |\psi\rangle + |\psi\rangle \otimes |\phi\rangle) \tag{1}$$

where N is the normalization factor. If $|\psi\rangle$, and $|\phi\rangle$ are properly normalized, we get

$$1 = \langle\psi\phi|\psi\phi\rangle = \frac{1}{N^2}(2 + 2|\langle\psi|\phi\rangle|^2) \tag{2}$$

Therefore,

$$N = \frac{1}{\sqrt{2\,(1 + |\langle\psi|\phi\rangle|^2)}}. \tag{3}$$

Problem 2

The total energy of two identical particle is the sum of each particle's energy. Therefore, we have

$$E_{sys} = E_{n_1 n_2} = \frac{\hbar^2\pi^2}{2mL^2}(n_1^2 + n_2^2) \tag{4}$$

where n_1, n_2 each denotes the state of corresponding particle. The only state that add up to $E_{sys} = \hbar^2\pi^2/(mL^2)$ is $n_1 = n_2 = 1$. The state vector for this is

$$|sys\rangle_b = |1\rangle \otimes |1\rangle \tag{5}$$

Here, the subscript b stands for boson and f for fermion. Note that there is no fermionic state because of the Pauli exclusion principle.To see this explicitly, let's try to construct antisymmetric state.

$$|sys\rangle_f = N(|1\rangle \otimes |1\rangle - |1\rangle \otimes |1\rangle) = 0 \tag{6}$$

Therefore, we can't have antisymmetric state with $(n_1, n_2) = (1, 1)$. If the total energy is $E_{sys} = 5\hbar^2\pi^2/(2mL^2)$, there are two possible cases, $(n_1, n_2) = (1, 2)$ and $(n_1, n_2) = (2, 1)$. Thus,

$$|sys'\rangle_b = \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle) \tag{7}$$

$$|sys'\rangle_f = \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle - |2\rangle \otimes |1\rangle) \tag{8}$$

Note that there are no 'cross' terms as in problem 1, because $|1\rangle$ and $|2\rangle$ are orthogonal.

Problem 3

(1) Let $|x_1, x_2\rangle$ be an eigenvector of P_{12} with eigenvalue λ. Then,

$$|x_1, x_2\rangle = P_{12}^2 |x_1, x_2\rangle = \lambda^2 |x_1, x_2\rangle \tag{9}$$

Here, the first identity comes from the fact that by exchanging twice, we end up with the original state. From the above equation, $\lambda^2 = 1$, which means P_{12} has eigenvalues ± 1.

(2) Let's expand the given eigenstate by the complete set of position eigenstates.

$$|\omega_1, \omega_2\rangle = \int dx_1 \int dx_2 |x_1, x_2\rangle\langle x_1, x_2|\omega_1, \omega_2\rangle \tag{10}$$

Now, act P_{12} on this expression to get

$$P_{12}|\omega_1, \omega_2\rangle = \int dx_1 \int dx_2 P_{12}|x_1, x_2\rangle\langle x_1, x_2|\omega_1, \omega_2\rangle = \int dx_1 \int dx_2 |x_2, x_1\rangle\langle x_1, x_2|\omega_1, \omega_2\rangle \tag{11}$$

$$= \int dx_1 \int dx_2 (|x_2\rangle \otimes |x_1\rangle)(\langle x_1| \otimes \langle x_2|)(|\omega_1\rangle \otimes |\omega_2\rangle) \tag{12}$$

$$= \int dx_1 \int dx_2 \left(|x_2\rangle \otimes |x_1\rangle\right) \langle x_1|\omega_1\rangle\langle x_2|\omega_2\rangle \tag{13}$$

$$= \left(\int dx_2 |x_2\rangle\langle x_2|\omega_2\rangle\right) \otimes \left(\int dx_1 |x_1\rangle\langle x_1|\omega_1\rangle\right) \tag{14}$$

$$= \int dx_1 \int dx_2 |x_2, x_1\rangle\langle x_2, x_1|\omega_2, \omega_1\rangle = |\omega_2, \omega_1\rangle \tag{15}$$

Here, when going to the fourth line we used the fact that braket is a number. Thus, when we act P_{12} on the symmetric or antisymmetric states, we get

$$P_{12}|\omega_1\omega_2, S\rangle = |\omega_1\omega_2, S\rangle \tag{16}$$

$$P_{12}|\omega_1\omega_2, A\rangle = -|\omega_1\omega_2, A\rangle \tag{17}$$

Here, A and S stands for antisymmetric and symmetric state respectively.

(3) To prove $P_{12}X_1P_{12} = X_2$, let's act the operator on the basis $|x_1, x_2\rangle$. To prove this, calculate

$$\begin{aligned} &\langle x_1', x_2'|P_{12}X_1P_{12}|x_1, x_2\rangle = \langle x_2', x_1'|X_1|x_2, x_1\rangle = \langle x_2', x_1'|x_2|x_2, x_1\rangle \\ &= x_2\delta(x_2' - x_2)\delta(x_1' - x_1) = x_2\,\langle x_1', x_2'|x_1, x_2\rangle = \langle x_1', x_2'|X_2|x_1, x_2\rangle \end{aligned} \tag{18}$$

So, we have $P_{12}X_1P_{12} = X_2$. Other identities can be proven in the same way. Note that it is easier to use momentum basis for the momentum operator P_1, P_2. To prove $P_{12}\Omega(X_1, P_1; X_2, P_2)P_{12} = \Omega(X_2, P_2; X_1, P_1)$, let's expand Ω by the power series.

$$\begin{aligned} \Omega(X_1, P_1; X_2, P_2) = &\sum_n (a_n X_1^n + b_n X_2^n + c_n P_1^n + d_n P_2^n) \\ &+ \sum \text{(terms with two operators)} \\ &+ \sum \text{(term with three operators)} \\ &+ \sum_{n_1 n_2 n_3 n_4} a_{n_1 n_2 n_3 n_4} X_1^{n_1} X_2^{n_2} P_1^{n_3} P_2^{n_4} + \cdots \end{aligned} \tag{19}$$

Here, we can use various commutation relations to make the form as above. Now the task is reduced to proving the identity for arbitrary monomial. For example, let's prove the identity for the four operator terms.

$$\begin{aligned} P_{12}(X_1^{n_1}X_2^{n_2}P_1^{n_3}P_2^{n_4})P_{12} &= (P_{12}X_1^{n_1}P_{12})(P_{12}X_2^{n_2}P_{12})(P_{12}P_1^{n_3}P_{12})(P_{12}P_2^{n_4}P_{12}) \\ &= (P_{12}X_1P_{12}P_{12}X_1P_{12}\cdots P_{12})\cdots \\ &= (P_{12}X_1P_{12})^{n_1}(P_12X_2P_12)^{n_2}(P_12P_1P_12)^{n_3}(P_12P_2P_12)^{n_4} \\ &= X_2^{n_1}X_1^{n_2}P_2^{n_3}P_1^{n_4} \end{aligned} \tag{20}$$

Here, we inserted $P_{12}^2 = 1$ between each operators. We can proceed for other types of operators in a similar manner. Therefore,

$$P_{12}\Omega(X_1, P_1; X_2, P_2)P_{12} = \Omega(P_{12}X_1P_{12}, P_{12}X_1P_{12}; P_{12}P_1P_{12}, P_2P_{12}) = \Omega(X_2, P_2; X_1, P_1). \tag{21}$$

So the identity is proved.

(4) For two identical particles, the Hamiltonian is symmetric under exchange of two particles, which means that it is invariant under change of x_1, p_1 and x_2, p_2. From the result of (3), $P_{12}HP_{12} = H$. Since $U = e^{-iHt/\hbar}$, $P_{12}UP_{12} = U$. Suppose $|\psi\rangle$ is an eigenstate of P_{12} with eigenvalue λ. Then, this implies

$$P_{12}U|\psi\rangle = P_{12}UP_{12}P_{12}|\psi\rangle = UP_{12}|\psi\rangle = \lambda U|\psi\rangle \tag{22}$$

Thus, $U|\psi\rangle$ is also an eigenvector of P_{12} with the same eigenvalue λ. Therefore, the eigenstates remains as an eigenstate with the same eigenvalue as time flows.

Problem 4

(a) Assuming $x' = u$, $y' = v$, we need to solve for x and y in the equations

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{23}$$

Inverting, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{24}$$

Note that changing basis is just opposite to the rotation of the point.

Therefore,

$$T|x = u, y = v\rangle = |x' = u, y' = v\rangle = |x = u\cos\theta - v\sin\theta, y = u\sin\theta + v\cos\theta\rangle \tag{25}$$

(*c.f.*, Eqn. 9.107 of lecture notes) As an example, the state localized at $|x' = 1, y' = 0\rangle$, along the $+x'$ axis, is the result of transforming the state $|x = 1, y = 0\rangle$, and is the same state as the state $|x = \cos\theta, y = \sin\theta\rangle$.

(b) Let's calculate the matrix element of X' in the unprimed basis, along the lines of what we did in Eqns. 9.114-9.117 of the lecture notes:

$$\begin{aligned} &\langle x = u_1, y = v_1|X'|x = u_2, y = v_2\rangle \qquad (26) \\ &= \langle x = u_1, y = v_1|TXT^\dagger|x = u_2, y = v_2\rangle \\ &= \langle x = u_1\cos\theta + v_1\sin\theta, y = -u_1\sin\theta + v_1\cos\theta|X|x = u_2\cos\theta + v_2\sin\theta, -u_2\sin\theta + v_2\cos\theta\rangle \\ &= (u_2\cos\theta + v_2\sin\theta)\langle x = u_1, y = v_1|TT^\dagger|x = u_2, y = v_2\rangle \\ &= (u_2\cos\theta + v_2\sin\theta)\delta(u_1 - u_2)\delta(v_1 - v_2) \qquad (27) \end{aligned}$$

Note the step from the second line to the third: the rotation is by $-\theta$ instead of θ because we have $T^\dagger|x = u_2, y = v_2\rangle$, not $T|x = u_2, y = v_2\rangle$, and similarly for $\langle x = u_1, y = v_1|T$.

On the other hand,

$$\langle x = u_1, y = v_1|(X\cos\theta + Y\sin\theta)|x = u_2, y = v_2\rangle = (u_2\cos\theta + v_2\sin\theta)\delta(u_1 - u_2)\delta(v_1 - v_2) \tag{28}$$

Therefore, we have

$$X' = X\cos\theta + Y\sin\theta \tag{29}$$

The transformation rules for other operators can be also similarly calculated:

$$Y' = X\sin\theta - Y\cos\theta \tag{30}$$
$$P_x' = P_x\cos\theta + P_y\sin\theta \tag{31}$$
$$P_y' = P_x\sin\theta - P_y\cos\theta \tag{32}$$

When calculating P_x', P_y', it is easier to use momentum basis.

(c) In terms of the primed basis, the untransformed state's position-basis representation (wavefunction) given by

$$\psi(x' = u, y' = v) = \langle x' = u, y' = v|\psi\rangle = \langle x = u\cos\theta - v\sin\theta, y = u\sin\theta + v\sin\theta|\psi\rangle \tag{33}$$

(*c.f.*, Eqns 9.109-9.110) of the lecture notes) Here, we used $|x' = u, y' = v\rangle = T|x = u, y = v\rangle = |x = u\cos\theta - v\sin\theta, y = u\sin\theta + v\cos\theta\rangle$. By plugging in the result of PS 7 for $(n_x, n_y) = (0,0), (1,0), (0,1)$, we obtain

$$\psi_{q',(0,0)}(x', y') = \phi_{(0,0)}(x'\cos\theta - y'\sin\theta, x'\sin\theta + y'\cos\theta) = Ne^{-\alpha(\rho')^2/2} \tag{34}$$

$$\psi_{q',(1,0)}(x', y') = \phi_{(1,0)}(x'\cos\theta - y'\sin\theta, x'\sin\theta + y'\cos\theta) = N\rho'\cos(\phi' + \theta)e^{-\alpha(\rho')^2/2} \tag{35}$$

$$\psi_{q',(0,1)}(x', y') = \phi_{(0,1)}(x'\cos\theta - y'\sin\theta, x'\sin\theta + y'\cos\theta) = N\rho'\sin(\phi' + \theta)e^{-\alpha(\rho')^2/2} \tag{36}$$

where N is the normalization constant, we use the ${}_q'$ subscript to indicate that this is a representation in terms of the primed-coordinate position-basis elements, we used $x' = \rho'\cos\phi'$ and $y' = \rho'\sin\phi'$, and we use $\phi_{(n,m)}$ to denote the (coordinate-system-independent) isotropic 2D SHO basis functions found in PS7. The formulae tell us that we obtain the value of the untransformed state $|\psi\rangle$ in the primed coordinates by adding θ to ϕ' and then applying our polar formulae. This makes sense, as $\phi' = 0$ corresponds to $\phi = \theta$. Remember, $|\psi\rangle$ is not changed by the transformation.

Now, let's consider the active rotation of the state. For the primed-system position-basis representation of the transformed state, unitarity tells us

$$\psi'_{q'}(x' = u, y' = v) = \langle x' = u, y' = v|\psi'\rangle = \langle x = u, y = v|T^\dagger T|\psi\rangle = \langle x = u, y = v|\psi\rangle \tag{37}$$

(*c.f.*, Eqn 9.111 of lecture notes) The wavefunction of the primed state in primed coordinates is the same as that of the unprimed state in unprimed coordinates, so

$$\psi'_{q',(0,0)}(x', y') = Ne^{-\alpha(\rho')^2/2} \tag{38}$$

$$\psi'_{q',(1,0)}(x', y') = N\rho'\cos\phi' e^{-\alpha(\rho')^2/2} \tag{39}$$

$$\psi'_{q',(0,1)}(x', y') = N\rho'\sin\phi' e^{-\alpha(\rho')^2/2} \tag{40}$$

We obtain the unprimed-system position-basis representation of the transformed state as follows:

$$\psi'_q(x = u, y = v) = \langle x = u, y = v|\psi'\rangle = \langle x = u, y = v|T|\psi\rangle \tag{41}$$

$$= \left(T^\dagger|x = u, y = v\rangle\right)^\dagger |\psi\rangle \tag{42}$$

$$= (|u\cos\theta + v\sin\theta, -u\sin\theta + v\cos\theta\rangle)^\dagger |\psi\rangle \tag{43}$$

$$= \langle u\cos\theta + v\sin\theta, -u\sin\theta + v\cos\theta|\psi\rangle \tag{44}$$

(*c.f.*, Eqns 9112-9.113 of lecture notes) Note that we used $-\theta$ for the angle here because we have $T^\dagger|x = u, y = v\rangle$, not $T|x = u, y = v\rangle$ (see our calculation of the matrix elements of X' in the unprimed basis) . Therefore, the wavefunctions are

$$\psi'_{q,(0,0)}(x, y) = \phi_{(0,0)}(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) = Ne^{-\alpha\rho^2/2} \tag{45}$$

$$\psi'_{q,(1,0)}(x, y) = \phi_{(1,0)}(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) = N\rho\cos(\phi - \theta)e^{-\alpha\rho^2/2} \tag{46}$$

$$\psi'_{q,(0,1)}(x, y) = \phi_{(0,1)}(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) = N\rho\sin(\phi - \theta)e^{-\alpha(\rho^2)/2} \tag{47}$$

where now the $_q$ subscript indicates we are calculating the unprimed-coordinate position-basis representation. These results make sense: the transformed state should be aligned with the x' and y' axes, so its value at $\phi = 0$ should be its value at an angle $|\theta|$ CW from the x' axis.

(d) For the anisotropic SHO, the Hamiltonian is given by

$$H = \frac{P_x^2 + P_y^2}{2m} + \frac{1}{2} m\omega_x^2 X^2 + \frac{1}{2} m\omega_y^2 Y^2 \tag{48}$$

where $\omega_x \neq \omega_y$. Under the transformation we obtained in (c), Hamiltonian transforms

$$\begin{aligned} H' = & \frac{(P_x \cos\theta - P_y \sin\theta)^2 + (P_x \sin\theta + P_y \cos\theta)^2}{2m} \\ & + \frac{1}{2} m\omega_x^2 (X\cos\theta - Y\sin\theta)^2 + \frac{1}{2} m\omega_y^2 (X\sin\theta + Y\cos\theta)^2 \end{aligned} \tag{49}$$

$$= \frac{P_x^2 + P_y^2}{2m} + \frac{1}{2} m\omega_x^2 X^2 + \frac{1}{2} m\omega_x^2 Y^2 + \frac{1}{2} m(\omega_y^2 - \omega_x^2)(X\sin\theta + Y\cos\theta)^2. \tag{50}$$

Therefore, Hamiltonian is not invariant under rotation if $\omega_x \neq \omega_y$.

(e) We want to find the generator of the symmetry transformation. As usual, let's use the relation between transformed and untransformed states under an infinitesimal transformation to figure this out (like Eqns 9.121-9.128 of the lecture notes). The infinitesimal rotation operator is

$$T(\delta\theta) = I - \frac{i}{\hbar}\,\delta\theta\, G \tag{51}$$

Let's calculate the position-basis representation of a transformed state, in the unprimed coordinates:

$$\langle x = u, y = v|T(\delta\theta)|\psi\rangle = \langle x = u, y = v|\psi\rangle - \delta\theta\,\frac{i}{\hbar}\langle x = u, y = v|G\,|\psi\rangle \tag{52}$$

$$= \psi_q(x = u, y = b) - \delta\theta\,\frac{i}{\hbar}\langle x = u, y = v|G\,|\psi\rangle \tag{53}$$

Let's also let the transformation operator act to the left on the basis element:

$$\langle x = u, y = v|T(\delta\theta)|\psi\rangle \tag{54}$$

$$= \left(T^\dagger(\delta\theta)|x = u, y = v\rangle\right)^\dagger |\psi\rangle \tag{55}$$

$$= (|x = u\cos\delta\theta + v\sin\delta\theta, -u\sin\delta\theta + v\cos\delta\theta\rangle)^\dagger\,|\psi\rangle \tag{56}$$

$$= \langle x = u\cos\delta\theta + v\sin\delta\theta, -u\sin\delta\theta + v\cos\delta\theta|\psi\rangle \tag{57}$$

$$= \psi(x = u\cos\delta\theta + v\sin\delta\theta, -u\sin\delta\theta + v\cos\delta\theta) \tag{58}$$

$$\approx \psi_q(x = u + v\delta\theta, v - u\delta\theta) \tag{59}$$

$$= \psi_q(x = u, y = v) + v\,\delta\theta\frac{\partial}{\partial x}\,\psi_q(x = u, y - v) - u\,\delta\theta\frac{\partial}{\partial y}\,\psi_q(x = u, y - v) \tag{60}$$

$$= \psi_q(x = u, y = v) - \delta\theta\,\frac{i}{\hbar}\,(\langle x = u, y = v|X\,P_y|\psi\rangle - \langle x = u, y = v|Y\,P_x|\psi\rangle) \tag{61}$$

$$= \psi_q(x = u, y = v) - \delta\theta\,\frac{i}{\hbar}\langle x = u, y = v|\,(X\,P_y - Y\,P_x)\,|\psi\rangle \tag{62}$$

where, in the second step, the sign of θ arises from the fact that we are acting with $T^\dagger$, not T. Equating the two expressions and noting that $\langle x = u, y = v|$ and $|\psi\rangle$ are arbitrary lets us conclude

$$G = X\,P_y - Y\,P_x \tag{63}$$

You will recognize this as the quantum analogue of the z-axis angular momentum, $l_z = x\,p_y - y\,p_x$. This operator's expectation value and matrix elements will therefore be conserved with time.

With the above explicit form for G, we may see explicitly that the generator is conserved:

$$[H, XP_y - YP_x] = X[H, P_y] + [H, X]P_y - Y[H, P_x] - [H, Y]P_y \tag{64}$$

$$= X[\frac{1}{2}m\omega^2Y^2, P_y] + [\frac{P_x^2}{2m}, X]P_y - Y[\frac{1}{2}m\omega^2x^2, P_x] - [\frac{P_y^2}{2m}, Y]P_y \tag{65}$$

$$= \frac{1}{2}m\omega^2(X2i\hbar Y - Y2i\hbar X) + \frac{1}{2m}(-2i\hbar P_xP_y + 2i\hbar P_yP_x) = 0. \tag{66}$$

Problem 5

Let's write the Lagrangian as

$$L = \frac{1}{2}m\dot{x}^2 - V(x) \tag{67}$$

Then, the Hamiltonian is

$$H = p\dot{x} - L = \frac{p^2}{2m} + V(x) \tag{68}$$

where $p = \frac{\partial L}{\partial \dot{x}}$. Using the Ehrenfest theorem, we get

$$\frac{d}{dt}\langle\frac{\partial L}{\partial \dot{x}}\rangle = \frac{d}{dt}\langle p\rangle = \frac{1}{i\hbar}\langle[p, H]\rangle = \frac{1}{i\hbar}\langle[p, V(x)]\rangle = \frac{1}{i\hbar}\langle(-i\hbar)V'(x)\rangle = -\langle\frac{dV}{dx}\rangle = \langle\frac{\partial L}{\partial x}\rangle \tag{69}$$

Thus, we have the Euler-Lagrange equation.