Ay123

Fall 2013

STELLAR STRUCTURE AND EVOLUTION

FINAL EXAM

Due Monday, December 16th 2013 at 9.00 am

This is a **closed book exam**. No notes (not even your lecture notes) or consultation. An unprogrammeable handheld calculator is acceptable, if you need it. Please take three straight hours in a quiet, isolated, location and do not look at the problems until you are ready to begin. All questions carry equal weight.

Please provide clear textual explanations inbetween your equations! We will penalize solutions whose logic can't be followed.

Some possibly useful constants are provided at the end of the questions.

Please hand in your completed paper to Judy McClain **in person** by Monday December 16th at 9.00 am. This is a hard deadline please!

Attempt FIVE questions

1. For an assumed equation of state, a solution for the run of pressure, temperature and density can be obtained for a star in hydrostatic equilibrium. Show that, for a polytropic equation of state:

$$P = K\,\rho^{\gamma}$$

where K and γ are independent of radius, the density dependence is given by the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\,(\xi^2\,\frac{d\theta}{d\xi}) = -\theta^n$$

where the density and radius are expressed in terms of dimensionless variables via $\rho = \lambda\,\theta^n$, the radius $r = \alpha\,\xi$, respectively and the index $n = (\gamma - 1)^{-1}$.

Accordingly, show that the gravitational potential energy of a star is then given by the expression

$$\Omega = -\frac{3\,G\,M^2}{(5\,-\,n)R}$$

What is the implication of this result for the range of n applicable to stable stars. Summarize the situations during stellar evolution for which polytropic solutions are appropriate, explaining clearly the value of n that is applicable in each case.

2. By considering the stability of a parcel of gas which expands and contracts adiabatically, show using diagrams and relevant text to explain the physical assumptions, that the condition for stability against convection in a star obeying the ideal gas law is:

$$\frac{dT}{dr} > \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \frac{dP}{dr}$$

Use the relevant stellar structure equations for a radiative star to show that this reduces to:

$$L(< r) < \left(1 - \frac{1}{\gamma}\right) \frac{16\pi \, acT^4 GM(r)}{3\kappa \, P}$$

where $L(r)$ and $M(r)$ are respectively the luminosity and enclosed mass at radius r, κ is the opacity and a is the radiation constant.

Thus show that to avoid convection in a stellar region where the equation of state is that of an ideal monatomic gas, the luminosity at a given radius must be limited by (in cgs units):

$$L(r) < 1.22 \times 10^{-18} \frac{\mu \, T^3}{\kappa \, \rho} M(r)$$

3. Show that, for the general case, the pressure in an electron gas is given by the integral of the kind

$$P = \frac{1}{3} \int_0^{P_F} v \, p \, f(p) \, dp = \frac{8\pi \, c}{3h^3} \int_0^{p_F} \frac{p \, / \, (m_e \, c)}{\sqrt{1 \, + \, p^2 \, / \, (m_e^2 \, c^2)}} \, p^3 \, dp$$

where m_e is the electron rest mass and p_F the Fermi momentum.

Now consider how this expression can be simplified in the non-relativistic and ultra-relativistic case. Derive the critical density, ρ_{crit}, for the transition between these two extremes assuming, for simplicity, that $n_e = \rho \, / m_H$ in determining the electron number density.

Explain, in simple terms (i.e. without further lengthy derivations), why the radius of a degenerate non-relativistic white dwarf becomes smaller as its mass increases and why, in the ultrarelativistic case, there is a maximum stable mass.

4. Describe, with the use of the Hertzsprung-Russell (H-R) diagram, how a protostar collapses onto the Main Sequence. Under what conditions does such a protostar satisfy hydrostatic equilibrium? Why are such young systems more luminous than those burning hydrogen on the Main Sequence?

Estimate the time it would take a large 1 $M_\odot$ proto-galactic cloud of near-uniform density to contract to a radius $R_\odot$ assuming nuclear reactions had not yet ignited, and that the star cooled via emission of blackbody radiation from its surface at a temperature $T_{\rm eff} = 2500$ K.

Why does such a star change the direction of its track on the H-R diagram prior to finally arriving at the Zero Age Main Sequence?

5. Calculate the mean thermal energy of a proton in the core of a star where hydrogen is burning at a temperature of 10^7 K. Estimate the closest distance such energy could bring two protons in proximity and compare this with the radius of each proton. Explain how this Coulomb barrier is overcome and (without any lengthy derivation) why, for non-resonant reactions, there is a preferred energy at which fusion occurs.

Sketch how the binding energy/nucleon varies with atomic mass. Above temperatures of 3. 10^9 K, photons have sufficient energy to dissociate a heavy nucleus. Write down the expression for the relative abundances attained by species i, j and k where forward and backward reactions of the type $i + j \rightarrow k$ and $k \rightarrow i + j$ achieve statistical equilibrium.

Why does such a statistical equilibrium lead to the continued production of heavier elements at the time of Silicon burning, but as we proceed to the iron peak itself, the bias towards heavier products ends?

6. Describe the various ways in which solar neutrinos can be counted on Earth and why the different techniques complement one another. Why was it thought there was a solar neutrino problem and what solutions were originally proposed for its resolution. Now consider yourself a helioseismologist skeptical of the claims that there is something wrong with the solar model. What techniques and results would you use to make such a case? Finally, how did particle physicists resolve the solar neutrino discrepancy?

Explain what role neutrinos play in the collapse of a massive star prior to the production of a Type II supernova. By what mechanism are they produced? If the collapsing core contains 1.4 solar masses of Fe^{56}, estimate the total energy released in neutrinos.

7. Prove that the sound speed c_S for an isothermal gas of uniform density ρ is given by the expression:

$$c_S = \frac{\gamma\, k\, T}{\mu\, m_H}$$

where μ is the mean molecular weight. Provide an order of magnitude estimate of the value for the Sun. What is meant by the *scale height* H of the solar atmosphere?

Discuss qualitatively how pressure waves can propagate within the Sun provided their angular frequency ω exceeds a critical value $\omega_C = c_S/2H$. Estimate the value of ω_C just below the solar photosphere.

Summarize what has been learned from quantitative studies of pressure waves within the Sun.

8. (The Eddington Approximation) Assume a source function increasing linearly with optical depth, i.e.

$$S_\nu(\tau_\nu) = a_\nu + b_\nu \tau_\nu.$$

(a) Define optical depth τ, specific intensity I_ν and the source function S_ν for a radiation field in a gas. Explain what is meant by the Rosseland mean opacity and how it is used. Give dimensions for each of these variables.

(b) Integrate the equation of radiative transfer to find $I_\nu(\overline{\tau})$, where $\overline{\tau}$ is an appropriate mean over frequency of τ_ν.

(c) Show that the effective temperature T_{eff} is equal to $T(\overline{\tau} = 2/3)$.

(d) For the same source function, derive an expression for the second moment of the radiation field, $K_\nu = 0.5 \int_{-1}^{+1} I_\nu \mu^2 d\mu$, where $\mu = cos(\theta)$.

(e) What does your result from part (c) imply about the surface radiation pressure from photons of frequency ν, in particular its relationship to the source function at some characteristic optical depth? What is this characteristic optical depth?

Useful constants:

Gravitational constant $G = 6.6738 \times 10^{-11}$ m^3 kg^{-1} sec^{-2}
Proton mass: $m_P = 1.67262 \times 10^{-24}$ g
Proton radius: $r_P = 1.2 \times 10^{-13}$ cm
Electron mass: $m_e = 9.11 \times 10^{-28}$ g
Neutron mass: $M_N = 1.67493 \times 10^{-24}$ g
Radius of Fe^{56} nucleus $\simeq 3.\ 10^{-13}$ cm
Planck's constant: $h = 6.6 \times 10^{-27}$ ergs s
Solar mass: $M_\odot = 1.99 \times 10^{33}$ g
Solar luminosity: $L_\odot = 3.85 \times 10^{33}$ ergs sec^{-1}
Solar radius: $R_\odot = 6.96 \times 10^{10}$ cm
Boltzmann's constant: $k = 1.38 \times 10^{-16}$ erg deg^{-1}
Stefan-Boltzmann constant: $\sigma = 5.67 \times 10^{-5}$ ergs cm^{-2} deg^{-4} s^{-1}
Radiation density constant: $a = 7.56 \times 10^{-15}$ ergs cm^{-3} deg^{-4}
Energy conversion: 1 eV = 1.6×10^{-12} ergs