Ay123

Fall 2013

STELLAR STRUCTURE AND EVOLUTION

Problem Set 5

Solutions due Monday November 25th 2013

1. Consider a first order ordinary differential equation $dy/dx = f(x)$, where $f(x)$ is a known function which we wish to solve for $y(x)$. We seek a solution by numerical integration and will evaluate the error in predicting $y(x_i + h)$ from a known initial value $y(x_i)$.

   (a) Show that the error for a single step using the Euler scheme (replacing derivatives by differences) is proportional to h^2, where h is the step and the interval goes from 0 to 1.0.
   (b) Show that using a mid-point technique (i.e. a second order Runge-Kutta integration) makes the error proportional to h^3.

   Consider applying these techniques to the first order ordinary differential equation

   $$\frac{dy}{dx} = x^2 + sin(x)$$

   over the interval in x from 0 to 1. (x in radians.)

   (c) Solve this equation analytically and plot $y(x)$ over that interval.
   (d) Use a second-order Runge-Kutta numerical integration. Use 40 steps over the interval in x. Plot your result for $y(x)$. What is the largest error in y in absolute value and as a fraction of y ?

2. Consider a cool dense gas cloud containing material with solar composition, so that $X = 0.90$. Such a freely collapsing gas cloud can be considered a "protostar" when it is fully ionized, which occurs when

   $$\frac{R}{R_\odot} = \frac{43.2}{(1 - 0.2X)} \frac{M}{M_\odot}$$

   Denote this as time $t = 0$, with R_0 as the initial radius.
   A gravitationally collapsing star with no nuclear energy sources follows the Hayashi track defined by $\log L = 15 \log T_{eff} + 0.2 \log M + \delta$. The constant δ is determined by the fact that a solar mass star at $t = 0$ has $L/L_\odot = 40$ when its radius is R_0.
   The pre-main sequence star switches to follow a new track (the so-called 'Henyey' track) once it develops a radiative core. This is given by:

   $$\frac{L}{L_\odot} \propto (\frac{M}{M_\odot})^{5.5} \left(\frac{R_\odot}{R}\right)^{0.5}$$

   As the energy continues to be supplied by gravitational contraction, we use the virial theorem to find L, viz. $L(t) = \frac{1}{2} d\Omega/dt$, and $\Omega = - 0.4\, G\, M^2/R$.

We define the main sequence by:

$$L \propto M^3, \quad R \propto M^{(n-1)/(n+3)}$$

where n is the exponent of the energy generation rate, $\epsilon \propto \rho T^n$, $n = 4$ for the $p-p$ chain ($M < 2.0 M_\odot$) and 16 for the CNO cycle ($M > 2.0 M_\odot$).

Considering stars of mass from 0.5 to 16 $M_\odot$ in steps of a factor of 2 in mass:

(a) Determine the evolutionary track in the log L versus log T_{eff} plane, marking the point where a pre-main sequence star switches from the Hayashi track to the the Henyey track in order to reach the main sequence at its appropriate L and T_{eff}. Provide a table of the L, R and T_{eff} values at the switching point as a function of mass. Plot a H-R diagram with the evolutionary tracks for the pre-main sequence stars for each mass, indicating the main sequence.

(b) Using the relations given above, find the time that the pre-main sequence star spends on the Hayashi track, on the Henyey track, and the total time required to reach the main sequence as a function of mass.

3. Suppose the collapse of a protostar is halted when the temperature in a star reaches a critical value T_c required for hydrogen burning. Suppose also that all the initial gravitational energy goes into thermal energy. Show that the greater the mass of the star, the smaller the density at the point where T_c is reached:

$$\rho_c = \frac{3}{4\pi\, M^2} \left(\frac{kT_c}{m_H\, G} \right)^3$$

Noting the criterion for electron degeneracy, estimate the critical mass below which collapse is halted by electron degeneracy, not by hydrogen burning. Show that this mass is related to the Chandasekhar limit by the approximate relation:

$$\frac{M_{crit}}{M_{Ch}} = \mu_e^2\, \frac{3}{2}\, \frac{1}{\pi\, 5^{3/2}} \left(\frac{k\, T_c}{m_e\, c^2} \right)^{3/4}$$