

AY 123 Pset 3 Solutions

1. Ionization fraction: $\frac{n_I}{n_I + n_{II}} = \frac{1}{1 + \frac{n_{II}}{n_I}}$

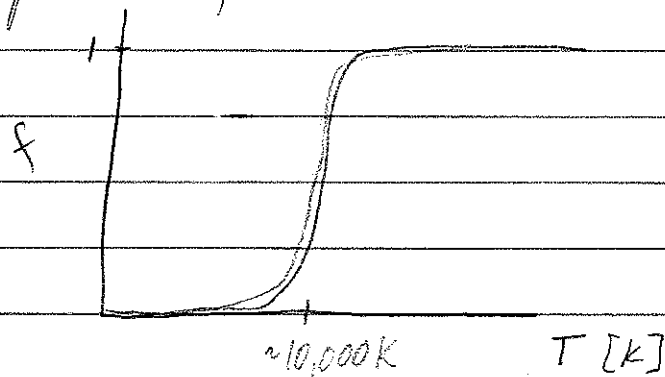
Ionization fraction is also given by: $y = \frac{n^+}{n} = \frac{n_e}{n}$

$$\frac{n_{II}}{n_I} = \frac{1}{n_e} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} \frac{g_{II}}{g_I} \exp\left(\frac{-\chi_I}{kT}\right)$$

$$\frac{n_{II}}{n_I} = \frac{1}{n_e} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} \exp\left(\frac{-\chi_I}{kT}\right) = \frac{y^2}{1-y}$$

$$n_e = \frac{\rho}{m_u}$$

Plot of ionization fraction vs T should look like:



Now solve for Γ_1 using the formalism presented in 3.7 of HKT.

$$P = P_0 \rho^{\chi_p} T^{\chi_T}, \text{ where } P_0, \chi_p, \chi_T \text{ are constants.}$$

$$\Rightarrow \chi_T = \left(\frac{d \ln P}{d \ln T} \right)_{(p \text{ or } \rho)} = \frac{T}{P} \left(\frac{dP}{dT} \right)_{(p \text{ or } \rho)}$$

$$1. \text{ (Cont.) } \chi_p = \left(\frac{d \ln P}{d \ln \rho} \right)_T = - \left(\frac{d \ln P}{d \ln V_p} \right)_T = \frac{\rho}{P} \left(\frac{\partial P}{\partial \rho} \right)_T = - \frac{1}{\rho P} \left(\frac{\partial P}{\partial V_p} \right)_T$$

$$\Gamma_1 = \left(\frac{d \ln P}{d \ln \rho} \right)_{ad} = - \left(\frac{d \ln P}{d \ln V_p} \right)_{ad}$$

$$\Gamma_1 = \chi_T (\Gamma_3 - 1) + \chi_p$$

$$\text{Where } \Gamma_3 - 1 = \frac{\rho}{P} \frac{\chi_T}{C_{v,p}} = \frac{1}{P} \left(\frac{\partial P}{\partial E} \right)_p$$

$$P = (n_e + n^+ + n^0) kT = (1+y) N \rho kT$$

specific internal energy.

$$E = (1+y) \frac{n}{\rho} \frac{3}{2} kT + y \frac{n}{\rho} \chi_H$$

$$= (1+y) N \frac{3}{2} kT + y N \chi_H$$

nuclei per gram

Now we have 3 relations: $P = P(\rho, T, y)$, $E = E(\rho, T, y)$, and the Saha eqn (pg. 1). Take the total differentials of the first two to find:

$$dP = P \left[\frac{dT}{T} + \frac{d\rho}{\rho} + \frac{dy}{1+y} \right]$$

$$\text{and } dE = \frac{3}{2} N k T (1+y) \left[\frac{dT}{T} + \frac{2}{3} \left(\frac{3}{2} + \frac{\chi_H}{kT} \right) \frac{dy}{1+y} \right]$$

$$1. \text{ (cont.) } \Gamma_3 - 1 = \frac{2 + 2D(y)}{3 + 2D(y)} \left(\frac{3}{2} + \frac{\gamma_H}{kT} \right)$$

From the eqn for Γ_1 , we get:

$$\Gamma_1 = \gamma_T (\Gamma_3 - 1) + \gamma_P$$

$$= \frac{[1 + D(y) \left(\frac{3}{2} + \frac{\gamma_H}{kT} \right)] [2 + 2D(y) \left(\frac{3}{2} + \frac{\gamma_H}{kT} \right)] + [1 - D(y)]}{3 + 2D(y) \left(\frac{3}{2} + \frac{\gamma_H}{kT} \right)^2}$$

$$\Gamma_1 = \frac{D(y) \left[2 \left(\frac{3}{2} + \frac{\gamma_H}{kT} \right)^2 + 4 \left(\frac{3}{2} + \frac{\gamma_H}{kT} \right) - 3 \right] + 5}{2D(y) \left(\frac{3}{2} + \frac{\gamma_H}{kT} \right)^2 + 3}$$

Adiabatic exponents \rightarrow The dimensionless adiabatic exponents (the Γ 's), measure the thermodynamic response of the system to adiabatic changes. \rightarrow constant entropy processes

For an ideal gas, $\Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma$

$$PV^{\Gamma_1} = C_1, \quad P^{1-\Gamma_2} T^{\Gamma_2} = C_2, \quad TV^{\Gamma_3-1} = C_3$$

where C_i are constants.

For more general eqns of state (comp. to ideal gas), Γ_1 , Γ_2 , and Γ_3 are distinct and carry information emphasizing different aspects of gas thermodynamics.

1. (cont) Take the differential of the Saha eqn and divide the result by the Saha eqn itself to find

$$\frac{dy}{1+y} = D(y) \left[\left(\frac{3 + \chi_H}{2} \right) \frac{dT}{T} - \frac{dp}{p} \right]$$

$$\text{Where } D(y) = \frac{y(1-y)}{(2-y)(1+y)}$$

$$dE = \frac{3}{2} N k T (1+y) \left\{ \left[1 + D \frac{2}{3} \left(\frac{3 + \chi_H}{2} \right)^2 \right] \frac{dT}{T} - D \frac{2}{3} \left(\frac{3 + \chi_H}{2} \right) \frac{dp}{p} \right\}$$

$$\frac{dP}{P} = \left[1 + D \left(\frac{3 + \chi_H}{2} \right) \right] \frac{dT}{T} + (1-D) \frac{dp}{p}$$

so that $\chi_p = 1 - D(y)$

$$\chi_T = 1 + D(y) \left(\frac{3 + \chi_H}{2} \right)$$

Note that $D(1) = D(0) = 0$ and, for general $0 \leq y \leq 1$, $D(y) \geq 0$. Since $D \geq 0$, we have $\chi_p \leq 1$ and $\chi_T \geq 1$. The interpretation is that if the temperature rises, keeping p fixed, we get more free e^- 's liberated and the pressure rises more so than the rise due to temperature alone.

Algebra
↓

2.a. To recreate the p - T phase diagram, look at each of the critical lines individually.

First look at the line at which the radiation and ideal gas pressure equate:

$$P = \frac{1}{3} a T^4 = \frac{\rho}{\mu m_u} kT$$

$$\rho = \frac{\mu m_u a T^3}{3k} \approx 1.5 \times 10^{-23} T^3 \text{ g cm}^{-3} \text{ K}^{-3} \quad (1)$$

Next, crystallization lines are determined by the factor

$$\Gamma_c \equiv \frac{(Ze)^2}{r_i kT} \sim \frac{\text{Coulomb energy}}{\text{Thermal energy}}$$

This value can be estimated using Lindemann's empirical rule, presented in Shapiro + Teukolsky's book chapter 4.3. This rough calculation yields $\Gamma_c \approx 75$, but experimental work has shown that ion quantum effects play a role and push the value up to ≈ 170

$$\Rightarrow \text{Substitute } \frac{1}{n_i} = \frac{\mu_i m}{\rho} = \frac{4\pi r_i^3}{3}$$

$$\rho = \frac{3 \mu_i m}{4\pi} \left(\frac{\Gamma_c kT}{(Ze)^2} \right)^3 = 8.49 \times 10^{-17} \Gamma_c^3 \mu_i Z^6 T^3 \quad (2)$$

Since Γ_c is defined as the ratio of Coulomb to Thermal energy, we

Can also use this result with $\Gamma_c \gg 1$ to find when Coulomb forces are important.

$$\rho = 8.49 \times 10^{-17} \Gamma_c^3 \mu_i Z^{-6} T^3 \quad (3)$$

Where $\Gamma_c = 1$

Next, consider pressure balance by degeneracy and Coulomb effects.

$$P_c = \frac{-3}{10} \left(\frac{4\pi}{3} \right)^{1/3} Z^{2/3} e^2 n_e^{4/3}$$

$$P_{e, nr} = \frac{3^{2/3} \pi^{4/3} \hbar^2}{5 m_e} n_e^{5/3}$$

Combining the two yields:

$$P_{tot} = P_{nr} \left[1 - \left(\frac{\rho_0}{\rho} \right)^{1/3} \right]$$

where $\rho_0 = \frac{Z^2 \mu_e m_p}{2 \pi^2 a_0^3} = 0.36 Z^2 \frac{\mu_e}{2} \text{ g cm}^{-3}$

Thermal ionization of H is given by the Saha eqn:

$$\frac{y^2}{1-y} = \frac{1}{n} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} \exp \left(\frac{-\chi_e}{kT} \right)$$

Setting $y = \frac{1}{2}$,

$$\rho_2 = 2m_p \left(\frac{2\pi m_e kT}{h^2} \right) \exp\left(-\frac{\chi_e}{kT}\right) \quad (4)$$

Pressure ionization sets in when the atoms' electron clouds begin to overlap:

$$r_i \approx a_0 = \frac{\hbar}{m_e e^2}$$

Writing r_i in terms of number density n :

$$\rho = \frac{3}{4\pi} m_H \left(\frac{m_e e^2}{\hbar^2} \right)^3 \approx 1.7 \text{ g cm}^{-3} \quad (5)$$

Finally, look at degeneracy conditions. Degeneracy is lifted when the Fermi energy of the electrons drops below the thermal energy, producing a large Maxwellian tail. The rough condition for this to occur is $E_F \approx kT$. So, for non-rel e^- :

$$kT \approx E_F = \frac{p_F^2}{2m_e} = 4.16 \times 10^{-11} \left(\frac{\rho}{\mu_e} \right)^{2/3}$$

$$\frac{\rho}{\mu_e} \approx 6.0 \times 10^{-9} T^{3/2} \quad (6)$$

Now we need to know when the e^- become relativistic to know where our non-rel approximation breaks down. This occurs when $p_f \approx m_e c$, so

$$\frac{\rho}{\mu_0} \approx 6.8 \times 10^5 \text{ g cm}^{-3}$$

Figure should be similar to that in HKT.

b. From part a, we had:

$$\rho_0 = \frac{Z^2 \mu_0 m_0}{2\pi^3 a_0^3} = 0.18 Z^2 \mu_0 \text{ g cm}^{-3}$$

↳ demarcation between Coulomb and degeneracy pressures.

This is within an order of magnitude of $\rho = \frac{m_0}{a_0^3}$ if $\mu_0 \approx 2$ and Z^2 is within an order of magnitude of π^3 . Most of the lighter elements satisfy this.

c. After looking up the densities and using $V = \frac{A_{\text{amp}}}{\rho}$ to

calculate the volume per atom, you see that the volumes per atom (or molecule of water), all agree within a factor of a few. The typical atomic/molecular separation is ≈ 3 Bohr radii.

2.c. (cont.)

Item	Density	Mass	Vol./atom	Spacing
Liquid H ₂	0.0708 g cm ⁻³	1 m _p	158 a ₀ ³	3.4 a ₀
Liquid He	0.125	4 m _p	359	4.4
Water	1.0	18 m _p	67.2	2.5
Iron	7.86	56 m _p	79.8	2.7

Problem 3

$$a) \langle \sigma v \rangle = 4\pi \left[\frac{m}{2\pi KT} \right]^{3/2} \int_0^{\infty} v \frac{S(E)}{E} \exp\left[-\frac{mv^2}{2KT}\right] \exp\left[-\frac{2\pi Z_A Z_B e^2}{\hbar v}\right] v^2 dv$$

First, re-write the integral in terms of E

$$v = \sqrt{\frac{2E}{m}}, \quad dv = \frac{dE}{\sqrt{2mE}}$$

$$\text{so } \langle \sigma v \rangle = 4\pi \left[\frac{m}{2\pi KT} \right]^{3/2} \int_0^{\infty} \sqrt{\frac{2E}{m}} \frac{S(E)}{E} \exp\left[-\frac{E}{KT}\right] \exp\left[-\frac{2\pi Z_A Z_B e^2}{\hbar \sqrt{\frac{2E}{m}}}\right] \frac{2E dE}{m \sqrt{2mE}}$$

cancelling terms and substituting $b = \frac{\pi \sqrt{2} Z_A Z_B e^2 \sqrt{m}}{\hbar}$ gives

$$\langle \sigma v \rangle = \sqrt{\frac{8}{\pi m}} \left[\frac{1}{KT} \right]^{3/2} S_0 \int_0^{\infty} \exp\left[-\frac{E}{KT} - \frac{b}{\sqrt{E}}\right] dE$$

where $S_0 \approx S(E)$ near the Gamow peak.

We want to approximate

$$(1) \left[\exp\left[-\frac{(E-E_0)^2}{(\Delta/2)^2}\right] \right] \approx \left(\frac{8}{\pi m}\right)^{1/2} (KT)^{-3/2} S_0 \exp\left[-\frac{E}{KT} - \frac{b}{\sqrt{E}}\right]$$

To find E_0 , match the peak location of the Gaussian (at E_0) with the peak location of the RHS.

The RHS peaks when $\frac{E}{KT} + \frac{b}{\sqrt{E}}$ is minimized.

$$\text{so } \frac{d}{dE} \left(\frac{E}{KT} + \frac{b}{\sqrt{E}} \right) \Big|_{E_0} = 0$$

$$\frac{1}{KT} - \frac{1}{2} \frac{b}{E_0^{3/2}} = 0$$

$$E_0 = \left(\frac{bKT}{2} \right)^{2/3}$$

Next, find C by matching the amplitudes at the peak.

At $E = E_0$, the amplitude of the LHS of (1) is simply C .

The amplitude of the RHS is

$$\left(\frac{8}{\pi m}\right)^{1/2} (kT)^{-3/2} S_0 \exp\left[-\frac{E_0}{kT} - \frac{b}{\sqrt{E_0}}\right]$$

so this is the value of C .

To find Δ , match the derivatives of (1) so that the functions are approximately equal.

$$\frac{d}{dx} \left((1) \right) \Big|_{E_0} = 0, \text{ since } E_0 \text{ is at a maximum.}$$

Match 2nd derivatives instead.

$$\frac{d^2}{dx^2} (\text{LHS}) \Big|_{E_0} = \frac{8C}{\Delta^2}$$

$$\begin{aligned} \frac{d^2}{dx^2} (\text{RHS}) \Big|_{E_0} &= \left(\frac{8}{\pi m}\right)^{1/2} (kT)^{-3/2} S_0 \exp\left[\frac{E_0}{kT} - \frac{b}{\sqrt{E_0}}\right] \left[\left(\frac{-1}{kT} + \frac{b}{2E_0^{3/2}}\right)^2 - \frac{3b}{4E_0^{5/2}} \right] \\ &= C \left(\frac{4E_0^3 - 4b kT E_0^{3/2} + b^2 k^2 T^2 - 3b k^2 T^2 E_0^{1/2}}{4k^2 T^2 E_0^3} \right) \end{aligned}$$

plugging in $E_0 = \left(\frac{b kT}{2}\right)^{2/3}$ and simplifying gives

$$\frac{8C}{\Delta^2} = \frac{3}{2^{1/3}} \frac{b}{(b kT)^{5/3}} C$$

$$\Delta = \left[\frac{2^{1/3} \cdot 8}{3} \frac{(b kT)^{2/3}}{b} \right]^{1/2}$$

with these definitions for C , E_0 , and Δ , we have

$$\langle \sigma v \rangle = C \int_0^{\infty} \exp\left[-\frac{(E - E_0)^2}{(\Delta/2)^2}\right] dE$$

$$= C \frac{\Delta}{2} \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$= \frac{\Delta}{2} C \sqrt{\pi}$$

$$b) \langle \sigma v \rangle = \frac{\Delta C}{\sqrt{\pi}} \propto \frac{1}{m^{1/2} T^{3/2}} S_0 \exp\left[-\frac{E_0}{KT} - \frac{b}{E_0^{1/2}}\right] \Delta$$

$\Delta \propto b^{1/3} T^{4/5}$ from part a, and $E_0 = \left(\frac{bKT}{2}\right)^{2/3}$

$$\text{so } \langle \sigma v \rangle \propto \frac{S_0 b^{1/3}}{m^{1/2} T^{2/3}} \exp\left[\frac{-b^{2/3}}{2^{2/3} (KT)^{1/3}} - \frac{2^{1/3} b^{2/3}}{(KT)^{1/3}}\right]$$

$$\propto \frac{S_0 b^{1/3}}{m^{1/2} T^{2/3}} \exp\left[\frac{-3b^{2/3}}{2^{2/3} (KT)^{1/3}}\right]$$

want $\langle \sigma v \rangle \propto e^{-\tau}$, so

$$\langle \sigma v \rangle \propto \frac{S_0 b^{1/3}}{m^{1/2} T^{2/3}} e^{-\tau} \propto \frac{S_0}{b m^{1/2}} \frac{b^{4/3}}{T^{2/3}} e^{-\tau}$$

$$\tau \propto \frac{b^{2/3}}{T^{1/3}}$$

$$\text{so } \langle \sigma v \rangle \propto \frac{S_0}{b m^{1/2}} \tau^2 e^{-\tau}$$

$$b \propto Z_A Z_B m^{1/2}$$

$$\text{so } \langle \sigma v \rangle \propto \frac{S_0}{Z_A Z_B m} \tau^2 e^{-\tau}$$

$$\tau = \frac{3b^{2/3}}{2^{2/3} (KT)^{1/3}}$$

Problem 4

a) From Problem Set 1,

$$\rho_c = \frac{3M}{4\pi R^3} = 5.57 \frac{M}{R^3} \text{ g/cm}^3 \text{ for } M, R \text{ in solar units}$$

$$T_c = \frac{56M}{12KR} = 5 \times 10^6 \frac{M}{R} \text{ K for } M, R \text{ in solar units}$$

assuming $R \propto M^{3/4}$ on the main sequence,

$$\rho_c = 5.57 M^{-5/4} \text{ g/cm}^3$$

$$T_c = 5 \times 10^6 M^{1/4} \text{ K}$$

Equation 6.76 of HKT gives

$$\epsilon_{pp} = \frac{2.4 \times 10^4 \rho X^2}{T_c^{2/3}} e^{-3.38/T_c^{1/3}} \text{ erg/g/s}$$

$$= \frac{2.4 \times 10^{10} \rho X^2}{T_c^{2/3}} e^{-3.38 \times 10^3/T^{1/3}} \text{ erg/g/s}$$

assuming solar abundances, $X=0.7$

then
$$\epsilon_{pp} = \frac{2.24 \times 10^6}{M^{17/12}} e^{-19.77/M^{1/2}} \text{ erg/g/s}$$

$$\epsilon_{cno} = \frac{4.4 \times 10^{25} \rho X Z}{T_c^{2/3}} e^{-15.228/T_c^{1/3}} \text{ erg/g/s}$$

$$= \frac{4.4 \times 10^{31} \rho X Z}{T_c^{2/3}} e^{-15.228 \times 10^3/T^{1/3}} \text{ erg/g/s}$$

$$= \frac{1.17 \times 10^{26}}{M^{17/12}} e^{-89.1/M^{1/2}} \text{ erg/g/s}$$

the total energy is just $\epsilon_{pp} + \epsilon_{cno}$

b) the fractional energy from each process is just

$$\frac{\epsilon_{pp}}{\epsilon_{tot}} = \frac{\epsilon_{pp}}{\epsilon_{pp} + \epsilon_{cno}}, \quad \frac{\epsilon_{cno}}{\epsilon_{tot}} = \frac{\epsilon_{cno}}{\epsilon_{pp} + \epsilon_{cno}}$$

Problem 5
 $\epsilon_{p+p} = \frac{QR}{\rho}$ where R is the reaction rate and Q is the energy released per reaction

so $\epsilon_{p+p} = \frac{Q}{\rho} \frac{1}{2} \frac{\rho^2}{m_p^2} \langle \sigma v \rangle X^2 \propto \rho$, so $n=1$

To find the temperature dependence of ϵ , use $\epsilon \propto \langle \sigma v \rangle \propto T^2 e^{-\tau}$

$$\epsilon \propto \rho T^n$$

$$\text{so } \rho T^n \propto T^2 e^{-\tau}$$

$$\text{so } \frac{d \ln(\rho T^n)}{d \ln T} = \frac{d \ln(T^2 e^{-\tau})}{d \ln T}$$

$$\eta = 2 \frac{d \ln T}{d \ln T} - \frac{d \tau}{d \ln T}$$

$$\tau = \frac{3b^{2/3}}{2^{2/3} (kT)^{1/3}} \quad \text{from problem 3}$$

$$\text{so } \frac{d \ln \tau}{d \ln T} = -\frac{1}{3}, \quad \frac{d \tau}{d \ln T} = -\frac{\tau}{3}$$

$$\boxed{\eta = \frac{\tau}{3} - \frac{2}{3}}$$

$$b = \frac{\pi \sqrt{2} Z_A Z_B e^2 \sqrt{m}}{\hbar}$$

for $p+p$, $Z_A = Z_B = 1$

$$m = \text{reduced mass} = \frac{m_p m_p}{m_p + m_p} = \frac{m_p}{2}$$

$$T_{co} = 1.5 \times 10^7 \text{ K}$$

$$\text{so } \tau_{opp} = 13.4 \quad \text{and} \quad \boxed{\eta \approx 4}$$

for ${}^7\text{Be} + p \rightarrow {}^8\text{B} + \gamma$, the same derivation gives $\eta = \frac{\tau}{3} - \frac{z}{3}$

now, $z_A = 4, z_B = 1$

$$m = \frac{m_p^7 m_p}{m_p + 7m_p} = \frac{7}{8}$$

everything else is identical to the $p + p$ reaction

$$\text{so } \tau = 42, \quad \eta \approx 14$$

Davis' experiment observed $\frac{1}{4}$ of the predicted neutrino flux

so a temperature change is needed such that

$$\frac{\text{obs rate}}{\text{predicted rate}} = \frac{1}{4} = \frac{T_{\text{new}}^{14}}{T_0^{14}}$$
$$\frac{T_{\text{new}}}{T_0} = \left(\frac{1}{4}\right)^{1/14} \approx 0.9$$

the Sun's core temperature would need to be 10% lower than predicted by models

such a change would require drastic changes to solar models

the factor of 4 discrepancy in neutrino flux is explained by the ability of electron neutrinos to change in μ and τ neutrinos, which could not be detected by Davis' experiment