AY 123 PSet 4 Solutions.

1. The scale height of an atmosphere is: $H = \frac{P}{g\rho}$

Assuming an ideal gas equation of state:

$$P = \frac{\rho N_a k T}{\mu}$$

Avogadro's cnst

$$\Rightarrow H = \frac{\cancel{\rho} k T N}{\mu} \frac{1}{g \cancel{\rho}} = \frac{N_a k T}{\mu g}$$

500 Km above the Sun, $g = \frac{G M_\odot}{R_\odot^2}$

$$H = \frac{N_a k T}{\mu} \frac{R_\odot^2}{G M_\odot} = 1.01 \times 10^7 \text{ cm} \cong \boxed{101 \text{ Km} = H}$$

Approximate $\omega_c \approx \frac{C_s}{2H}$ where $C_s^2 = \frac{\gamma P}{\rho}$.

$$\omega_c = \frac{1}{2H}\sqrt{\frac{\gamma P}{\rho}} = \frac{1}{2H}\sqrt{\gamma g H} = \frac{1}{2}\sqrt{\frac{\gamma g}{H}}$$

$\gamma = \frac{5}{3}$, $H = 101$ Km, $g = 2.7 \times 10^4$ cm/s^2

$$\Rightarrow \boxed{\omega_c = 0.0334 \text{ sec}^{-1}}$$

The corresponding period is 3 min

2.a. Consider a mass element displaced upwards in a star. The element will expand adiabatically. As it does so, its density will change by $\delta\rho$. The surrounding gas will have a density difference of $\Delta\rho$ from the element's initial position. The force per unit mass on the element will then be buoyancy minus gravity:

$$F = \frac{1}{\rho}\left(g\left[\rho+\Delta\rho\right] - g\left[\rho+\delta\rho\right]\right)$$

$$F = \frac{g}{\rho}\left(\Delta\rho - \delta\rho\right)$$

For small displacements, we can write this as:

$$F = -\omega_{BV}^2 \Delta x = \frac{g}{\rho}\left(\frac{\Delta\rho}{\Delta x} - \frac{\delta\rho}{\Delta x}\right)\Delta x$$

where $$\omega_{BV}^2 = -\frac{g}{\rho}\left(\frac{\Delta\rho}{\Delta x} - \frac{\delta\rho}{\Delta x}\right)$$ ⇐ Brunt-Vaisala frequency

b. If we consider the motion to be adiabatic:

$$P \sim \rho^\gamma$$

$$\frac{\delta\rho}{\rho} = \frac{1}{\gamma}\frac{dP}{P}$$

If we assume the surrounding gas is ideal, then:

$$P \propto \rho T$$

2b. (cont.): $\frac{\Delta\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T}$

Plug these into the eqn for ω_{BV}^2:

$$\omega_{BV}^2 = -g\left(\frac{1}{P}\frac{dP}{dx} - \frac{1}{T}\frac{dT}{dx} - \frac{1}{\gamma}\frac{dP}{P}\right)$$

$$\omega_{BV}^2 = -g\left(\frac{\gamma-1}{\gamma}\frac{1}{P}\frac{dP}{dx} - \frac{1}{T}\frac{dT}{dx}\right) \qquad (*)$$

Since an adiabatic temperature gradient will satisfy

$$PT^{\gamma/(1-\gamma)} = \text{const}$$

$$\rightarrow \frac{dT}{T} = \frac{\gamma-1}{\gamma}\frac{dP}{P}$$

Plugging this into $(*)$:

$$\omega_{BV}^2 = -g\left(\frac{\gamma-1}{\gamma}\frac{1}{P}\frac{dP}{dx} - \left[\frac{\gamma-1}{\gamma}\frac{1}{P}\frac{dP}{dx}\right]\right) = 0$$

This shows that gravity waves driven by buoyancy oscillations can't propagate in convective regions of stars. This is not surprising $\rightarrow$ buoyancy waves require displacements to feel a restoring force, while convection requires displacements to grow.

2.c. $\omega_{BV}^2 = -g\left(\frac{\gamma-1}{\gamma}\frac{1}{P}\frac{dP}{dx} - \frac{1}{T}\frac{dT}{dx}\right)$

First assume an isothermal atmosphere, $\frac{dT}{dz} = 0$

$$\Rightarrow \omega_{BV}^2 = -g\left(\frac{\gamma-1}{\gamma}\frac{1}{P}\frac{dP}{dz}\right)$$

From hydrostatic equilibrium, $\frac{dP}{dz} = -g\rho$

$$\omega_{BV}^2 = g^2\,\frac{\gamma-1}{\gamma}\,\frac{\rho}{P}$$

So $g = \frac{GM}{R^2}$ and $P = \frac{\rho k T}{\mu m_a}$

$$\omega_{BV}^2 = \frac{G^2M^2}{R^4}\,\frac{\gamma-1}{\gamma}\,\frac{\mu m_a}{kT}$$

For Earth's atmosphere, assume 80% N_2 and 21% O_2 $\Rightarrow \mu = 28.8$. Since the molecules have three spatial and two rotational degrees, $\gamma = \frac{7}{5}$. A typical temperature is 288 K.

The Martian atmosphere is mostly CO_2, so $\mu = 44$. CO_2 also has 3 spatial and 2 rotational degrees of freedom, so $\gamma = \frac{7}{5}$.

2.c. (cont.) Plugging these #s in:

$\omega_{BV,\odot} = 0.0182\,Hz$

$\omega_{BV,M} = 0.0093\,Hz$

These correspond to periods of 5.75 min & 11.3 min respectively

$$\omega_c^2 \simeq \frac{c_s^2}{4\rho H^2} \sim \frac{\gamma P}{4\rho H^2}$$

Since $H = \frac{P}{g\rho}$ for an ideal gas:

$$\omega_c^2 = \frac{\gamma \rho}{4 P g^2} = \frac{1}{4}\frac{\gamma^2}{\gamma - 1}\omega_{BV}^2$$

$\omega_{c,\odot} = 0.0201\,Hz \quad P = 5.20$ min

A wave between ω_c and ω_{BV} will decay instead of propagating.

3.a. $\frac{dP}{dr} = -g\rho$

Using the polytropic eqn of state and sound speed:

$P \propto \rho^{\gamma} \propto (c^2)^{\gamma/(\gamma-1)}$

$\frac{dP}{P} = \frac{\gamma}{\gamma-1} \frac{dc^2}{c^2}$

Plugging this into hydrostatic equilibrium:

$\frac{dc^2}{dr} \frac{1}{c^2} \left(\frac{\gamma}{\gamma-1}\right) P = -g\rho$

$\frac{1}{c^2} \frac{dc^2}{dr} = -\frac{(\gamma-1)}{\gamma} \frac{g\rho}{P}$

But we know $c^2 = \frac{\gamma P}{\rho}$

$\Rightarrow \frac{\cancel{\rho}}{\cancel{\gamma} \cancel{P}} \frac{dc^2}{dr} = -\frac{(\gamma-1)}{\cancel{\gamma}} \frac{g\cancel{\rho}}{\cancel{P}}$

$\frac{dc^2}{dr} = -(\gamma-1)g$

Since γ and g are both constant near the surface

3.a. (cont.) $\boxed{c^2 = (\gamma - 1) g z}$

b. The distance a p-wave can propagate into a star, given a fixed k_y and ω is found when $k_x = 0$ in the p-wave solution:

$$\omega^2 = c^2(z_{max}) k_y^2 \quad \text{[from } v = \nu\lambda \text{]}$$

$$\omega^2 = c^2 k_y^2$$

$$\omega^2 = (\gamma - 1) g z_{max} k_y^2$$

$$z_{max} = \frac{\omega^2}{k_y^2 (\gamma - 1) g}$$

$$k_y^2 = \frac{l(l+1)}{R^2}$$

$$\Rightarrow z_{max} = \frac{R^2 \omega^2}{l(l+1)(\gamma-1) g}$$

c. The n, l, + ω relation for standing waves is found by performing the following integral:

$$n\pi = \int_0^{z_{max}} k_r \, dr$$

k_r is found by rearranging the p-wave eqn. and solving for k_x:

$$n\pi = \int_0^{z_{max}} \sqrt{\frac{\omega^2}{(\gamma-1) g z} - \frac{l(l+1)}{R^2}} \, dr = \frac{\omega^2}{(\gamma-1) g} \frac{R}{\sqrt{l(l+1)}} \int_0^1 \sqrt{\frac{1}{x} - 1} \, dx$$

3.c. (cont.) Evaluating using Wolfram alpha, you get

$$\omega^2 = 2n(\gamma-1)g\frac{\sqrt{l(l+1)}}{R}$$

Problem 4

Start with equations 8.11 and 8.12 in HKT, which describe adiabatic radial oscillations:

$$\frac{d\zeta}{dr} = \frac{-1}{r}\left(3\zeta + \frac{1}{\gamma}\frac{\delta P}{P}\right) \qquad (8.11)$$

$$\frac{d}{dr}\left(\frac{\delta P}{P}\right) = -\frac{d\ln P}{dr}\left[4\zeta + \frac{\omega^2 r^3}{GM(r)}\zeta + \frac{\delta P}{P}\right] \qquad (8.12)$$

where $\zeta = \frac{\delta r}{r}$

plug in $\zeta =$ constant $\left(\frac{d\zeta}{dr} = 0\right)$

$$0 = \frac{-1}{r}\left(3\zeta + \frac{1}{\gamma}\frac{\delta P}{P}\right)$$

$$-3\zeta = \frac{1}{\gamma}\frac{\delta P}{P} \qquad (1)$$

Since ζ is constant, $\frac{\delta P}{P}$ is also constant.

Using this in (8.12) gives

$$0 = -\frac{d\ln P}{dr}\left[4\zeta + \frac{\omega^2 r^3}{GM(r)}\zeta + \frac{\delta P}{P}\right]$$

$$4\zeta + \frac{\omega^2 r^3}{GM(r)}\zeta + \frac{\delta P}{P} = 0 \qquad (2)$$

(1) and (2) are in fact the boundary conditions of (8.11) and (8.12) at $r=0$ and $r=R$, respectively. If $\frac{\delta r}{r}$ is constant, then these conditions apply everywhere.

Applying (1) and (2) at $r=R$ gives

$$-3\zeta(R) = \frac{1}{\gamma}\frac{\delta P}{P}$$

$$\boxed{-3\frac{\delta R}{R} = \frac{1}{\gamma}\frac{\delta P}{P}}$$

This is one of the equations derived in class.
(2) yields the other:

$$4\frac{\delta R}{R} + \frac{\omega^2 R^3}{GM}\frac{\delta R}{R} + \frac{\delta P}{P} = 0$$

using the first equation from class:

$$4\frac{\delta R}{R} + \frac{\omega^2 R^3}{GM}\frac{\delta R}{R} - 3\gamma\frac{\delta R}{R} = 0$$

$$(3\gamma-4)\frac{GM}{R^3} = \omega^2$$

$$\boxed{(3\gamma-4)\frac{4\pi}{3}G\bar{\rho} = \omega^2}$$

So, if $\frac{\delta r}{r}$ is constant, the equations from class are satisfied.

To find the period, use $P = \frac{2\pi}{\omega}$.

For $\gamma = 5/3$, $\log\left(\frac{M}{M_\odot}\right) = 0.8$, and $\log\left(\frac{R}{R_\odot}\right) = 1.4$

$$\omega = 1.25 \times 10^{-5}\ s^{-1}$$

$$P = 5 \times 10^5\ s = \boxed{5.8 \text{ days}}$$

To find the velocity, assume $\frac{\delta r}{r}$ follows sinusoidal variations.

$$\frac{\delta r}{r}(r,t) = \frac{\delta r}{r}(r)\sin\omega t$$

$$\delta r(r,t) = \frac{\delta r}{r_o} r \sin\omega t$$

so $v = \frac{\delta r}{r_o} r\omega\cos\omega t$

$$v_{max} = \frac{\delta r}{r_o} R\omega \quad \text{at } r = R$$

$$v_{max} = 2.19 \times 10^6 \text{ cm/s} = \boxed{21.9 \text{ km/s}}$$

for $\frac{\delta r}{r_0} = 0.1$

Assume T_{eff} changes because the temperature of the gas changes. This is an adiabatic process, so

$$\frac{\delta T_{eff}}{T_{eff}} = \frac{\delta T}{T} = (\gamma - 1)\frac{\delta \rho}{\rho} = \frac{\gamma - 1}{\gamma}\frac{\delta P}{P} = -(\gamma-1) 3 \frac{\delta r}{r}$$

for $\gamma = 5/3$,

$$\frac{\delta T_{eff}}{T_{eff}} = -2 \frac{\delta r}{r_0} \sin \omega t$$

so for $\frac{\delta r}{r_0} = 0.1$, $\frac{\delta T_{eff}}{T_{eff}}$ ranges between $\boxed{\pm 0.2}$

Problem 5

Use homology with CNO energy generation and gas pressure to find a relation between L and ρ. Since Cepheids are evolved and cooler than main sequence stars of the same mass use Kramer's opacity - $n = 17$, $\lambda = 2$, $\nu = 6.5$ gives

$$\bar{\rho} \propto M^{-14/13}$$
$$L \propto M^{67/13}$$
$$\text{so} \quad \bar{\rho} \propto L^{-14/67}$$

From problem 4, $\omega^2 \propto \bar{\rho}$

$$\text{so} \quad P \propto \bar{\rho}^{-1/2}$$

and

$$\boxed{P \propto L^{7/67}}$$

The constant can be found from observations. Find Cepheids in clusters with known distances from main sequence fitting. Measure brightness and period, then use brightness and distance to get luminosity. This can be done for many Cepheids to reduce uncertainties.