Ay126: Homework 3

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[1] Far Infrared Lines: Literature survey. Review the literature and write a short report on the highest redshift detections of various far infrared fine structure lines. Compare the luminosities of those detections with the luminosity of the Milky Way in the same lines. [10 points]

[2] Hyperfine Lines. Review the literature. List fine structure lines of the four most abundant elements (including isotopes, if it make sense) and their approximate frequencies and A coefficients. [10 points]

[3] Hyperfine Splitting. In the class we discussed the proper formula for the field from a magnetic dipole. The interaction of the proton ($\boldsymbol{\mu_p}$) and electron dipole moments ($\boldsymbol{\mu_e}$) results in an additional perturbing term for the Hamiltonian, H':

$$H' = -\frac{\mu_0}{4\pi}\frac{1}{r^3}\Big[3(\boldsymbol{\mu_p}\cdot\hat{r})(\boldsymbol{\mu_e}\cdot\hat{r}) - \boldsymbol{\mu_p}\cdot\boldsymbol{\mu_e}\Big] - \frac{2}{3}\mu_0\boldsymbol{\mu_p}\cdot\boldsymbol{\mu_e}\delta^3(\mathbf{r}). \tag{1}$$

This interaction term will perturb the energy of the ground state of the of H I by

$$E' = \int \psi_0^* H' \psi_0 dV \tag{2}$$

where ψ_0 is the $1s^1$ wave function and $dV = dxdydz$.

[**A**] Show that for any wave function which is purely radial, as is the case with any $n, l = 0$ state, the first term integrates to zero. [15 points]

[**B**]. Consult the notes or a standard book on QM for the expression of the $^1S_{1/2}$ state of hydrogen. Show that integration over the second term yields

$$E' = -\frac{2}{3}\mu_0\boldsymbol{\mu_p}\cdot\boldsymbol{\mu_e}|\psi_0(0)^2| \tag{3}$$

where a_0 is the Bohr radius. [15 points]

[**C**] Next, we relate the magnetic moments to the spin momenta as follows: $\boldsymbol{\mu_p} = \gamma_p \mathbf{S}_p$ and $\boldsymbol{\mu_e} = \gamma_e \mathbf{S}_e$ with γ_p and γ_e being the gyromagnetic ratios of the electron and proton respectively. Thus, the above equation simplifies to

$$E' \propto \langle \mathbf{S_e} \cdot \mathbf{S_p} \rangle \tag{4}$$

From QM we know that

$$\langle \mathbf{S}_e \cdot \mathbf{S}_p \rangle = \frac{1}{2} \Big\langle \mathbf{J} \cdot \mathbf{J} - \mathbf{S_e} \cdot \mathbf{S_p} - \mathbf{S_p} \cdot \mathbf{S_p} \Big\rangle \tag{5}$$

where $\mathbf{J} = \mathbf{S_e} + \mathbf{S}_p$. The electron and proton have spin of $1/2$ and so the eigenvalue of $\mathbf{S_e} \cdot \mathbf{S_e}$ and $\mathbf{S_p} \cdot \mathbf{S_p}$ is $S(S+1)\hbar^2$ or $3/4\hbar^2$. Likewise the eigenvalue of $\mathbf{J} \cdot \mathbf{J} = J(J+1)\hbar^2$. Compute and plot energy levels of the ground and excited state (in eV units). Identify the spectroscopic term for the two levels. [10 points]

[**D**] Research the literature and write down the most accurate value for the 21-cm line. Compare this to your answer. Who else, other than astronomers, are interested in this transition? [10 points]

[**E**] Now replace the electron by a muon (which is also a lepton like electron but has a mass of 106 MeV). Compute the hyperfine splitting for this "atom". [10 points]

[**4**] ***l*-*v* diagrams.** The purpose of this purely pedagogical exercise is to help you get comfortable with *l*-*v* diagrams. We assume that gas clouds are on circular orbits. [Why is this a reasonable assumption?].

Consider a line-of-sight (los) starting from the Earth and along Galactic longitude l and latitude $b = 0$. The radial velocity of a cloud, under this assumption, is given by

$$v_r = R_0 \Big[\Omega(R) - \Omega_0 \Big] \sin(l). \tag{6}$$

Here R is the galacto-centric radius of the cloud, R_0 is the radius of the solar circle (the distance from the Sun to the center of Galaxy) and Ω_0 is the local angular speed. $R_0 = 8.5\,\mathrm{kpc}$ and $V_0 = R_0\Omega_0 = 220\,\mathrm{km\,s^{-1}}$. Assume that the rotation curve is flat, that is, $V(R) = V_0$.

1. Derive the result stated in Equation 6. [5 pts]

2. For $l = 45°$ plot the run of v_r (in $\mathrm{km\,s^{-1}}$) as a function of distance from us, d (kpc), all the way to edge of the H I disk (say 20 kpc). [5 pts]

3. Equation 6 offers a ready way to estimate distances to H II regions or giant molecular clouds (in the absence of other distance measures, which is almost always the case).

However, there are deviations in the velocity field of the Galaxy due to a triaxial bulge or spiral density waves.

As before we set $l = 45°$. We choose the following points: (i) the tangent point,[1] (ii) a point on the solar radius (where the los intersects the solar circle; $d \approx 12\,\mathrm{kpc}$) and say at (iii) $d = 20\,\mathrm{kpc}$. Perturb the velocity fields (to keep it simple, just the radial part) by $10\,\mathrm{km\,s^{-1}}$ and $30\,\mathrm{km\,s^{-1}}$ and derive the corresponding uncertainty in the inferred distances. [5 pts]

4. Inspect l-v diagrams in HI and CO that were presented in the class (April 27) . You will see deviations from that predicted by the formula. List the deviations and offer plausible explanations. [Feel free to discuss this particular problem with other colleagues and postdocs etc]. [10 points]

[**5**] **Inferring Column Density.** Consider two clouds with spin temperature and net column density of T_i and $N_H(i)$ for $i = 1, 2$. Assume no other motions other than that expected from thermal effects. A pulsar is conveniently located behind the two clouds and you are thus able to obtain an optical depth, $\tau(v)$ as well as the emission brightness spectrum, $T(v)$. Consider $N_H(1) = N_H(2)$ (with value that is indicative of typical conditions) and T_1 to be CNM and T_2 to be WNM. What are the pitfalls you will be making in deducing the total column density using the emission spectrum, $T(v)$ and $\tau(v)$? [15 points]

[1]For a given l there is a minimum galactocentric radius that can be reached. The tangent point is the intersection between circle and the line of sight.