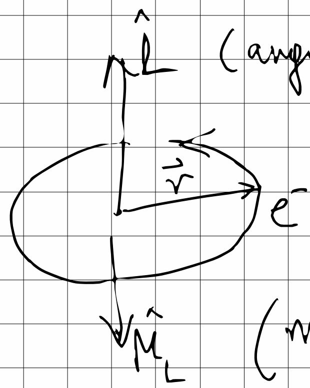
Spin-orbit interaction (& relativistic correction)



$\hat{L}$ (angular momentum)

$\hat{M}_L$ (magnetic dipole moment)

Current

$$i = \frac{ev}{2\pi r}$$

Ampere's Law:

$$M_L = i\pi r^2 = \frac{e}{2m_e} m_e v r = \frac{e}{2m_e} L$$

for historical reasons

$$M_L = \frac{e\hbar}{2m_e} g_L \frac{L}{\hbar}$$

where $\mu_B = \frac{e\hbar}{2m_e}$ is the Bohr magneton

$$= 5.8 \times 10^{-5} \text{ eV/Gauss}$$

where $g_L = 1$ (classically)

From QM

Modulus $$M_L = \frac{g_L \mu_B}{\hbar} \sqrt{\ell(\ell+1)}\,\hbar$$

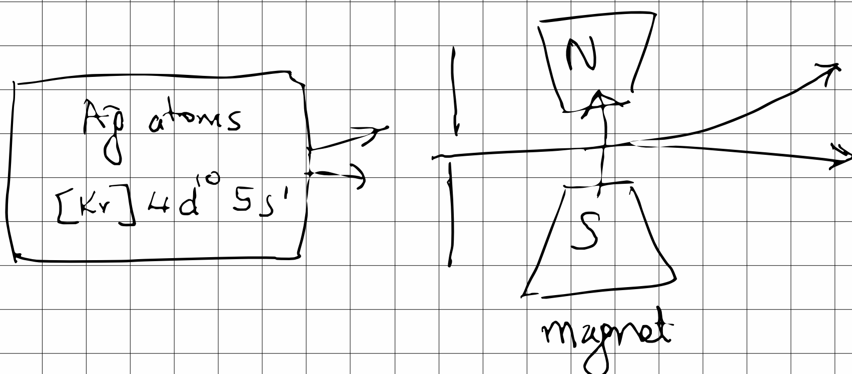
$$M_{L,z} = -g_L \mu_B m_\ell$$

There are $(2l+1)$ states for a given L.

From Schrodinger's solution

$$\text{number of states} = \sum 2l+1 = n^2$$

Stern–Gerlach experiment



Thus electron has spin

$$\vec{M}_s = -g_s \frac{\mu_B}{\hbar} \vec{S}$$

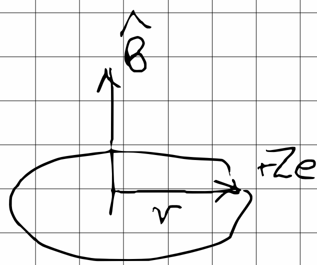
$$m_s = \pm \frac{1}{2} \qquad g_s = 2 \text{ (experimental)}$$

$$S^2 = s(s+1)\hbar^2$$

$$S_z = m_s \hbar$$

Consider a rotating frame centered on the electron.
The proton produces a magnetic field at the position of the electron

$$\vec{B} = \frac{1}{c}\frac{(-Ze\vec{v})\times r}{r^3}$$



where $-\vec{v}$ is the velocity of proton wrt electron

The electrostatic field is

$$\vec{E} = \frac{Ze}{r^3}\vec{r}$$

$$\vec{B} = \frac{1}{c}\frac{(-Ze\vec{v})\times\vec{r}}{r^3} = -\frac{1}{c^2}\vec{v}\times\vec{E}$$

But the electric field is the gradient of the electrostatic potential

$$e\vec{E} = -\vec{F} = \frac{dV(r)}{dr}\frac{\vec{r}}{r}$$

$$\vec{B} = -\frac{1}{c^2}\vec{v}\times\vec{E} = \frac{1}{em_ec^2}\frac{dV}{dr}\frac{1}{r}\vec{L}$$

We did the calculation in the frame of the electron. However, to compare the measurements, we need to calculate in the frame of proton. The transformation (complicated) shows the correct factor is $1/2$ smaller. (Thomas precession)

The ~~...~~ spin-orbit energy is

$$E_{SO} = -\frac{1}{2}\vec{M}_s \cdot \vec{B}$$

$$= \frac{1}{2m_e^2c^2}\frac{1}{r}\frac{dV(r)}{dr}\vec{S}\cdot\vec{L}$$

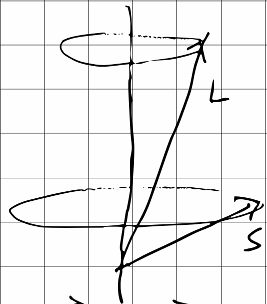
$$= \xi(r)\,\vec{S}\cdot\vec{L}$$

with $\xi(r) = \frac{1}{2m_e^2c^2}\frac{1}{r}\frac{dV(r)}{dr}$

For the Hydrogen atom $V(r) \propto 1/r^2$ and the Schrodinger solution gives the probability distribution of the electron

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{Z^3}{a_0^3 n^3 \, l(l+\frac{1}{2})(l+1)}$$

If $\vec{S}$ and $\vec{L}$ are independent, they would precess independently



However $\vec{S}$ and $\vec{L}$ are coupled (it so appears)

$$\vec{J} = \vec{L} + \vec{S}$$

In this case both $\vec{L}$ and $\vec{S}$ precess around $\vec{J}$

$$\therefore \; J^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S}$$

$$\vec{L}\cdot\vec{S} = \frac{\hbar^2}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]$$

$$E_{so} = \frac{\hbar^2}{2}\langle \xi(r) \rangle \left[j(j+1) - l(l+1) - s(s+1)\right]$$

The corresponding "term" value is

$$\Delta T = \frac{R\alpha^2 Z^4}{n^3 \ell(\ell+\frac{1}{2})(\ell+1)} \left(j(j+1) - \ell(\ell+1) - s(s+1) \right)$$

Consider a doublet $j = \ell \pm \frac{1}{2}$, $S=0$

$$\Delta T = \frac{R\alpha^2 Z^4}{n^3 \ell(\ell+1)}$$

$$= 5.82 \frac{Z^4}{n^3 \ell(\ell+1)} \quad cm^{-1}$$

Other relativistic effect:

- spin-orbit (spin arises from relativity)
- usual relativity $\frac{1}{2}mv^2 \to \gamma m c^2 - mc^2$
- Darwin contribution.

The electron position is uncertain $\frac{\hbar}{m_e c}$
So Coulomb interaction should be averaged over $\left(\frac{\hbar}{m_e c}\right)^3$

$$\Delta E_{rel} = E_n \frac{(Z\alpha)^2}{n^2} \left[\frac{n}{j+\frac{1}{2}} - \frac{3}{4} \right]$$

where E_n is the energy level of the classical system.