Ay 1 Problem Set 2
Due May 22

Do not agonize trying to be precise on these questions. They are intended as "back of the envelope" estimates.

Accretion Disks - 15 pts

We have learned that the process of fusion has an energy generation efficiency 0.7% – given material of mass m, it creates $0.007mc^2$ in energy. Now we will see how efficient accretion onto a black hole is...

1) It turns out that for a non-rotating black hole, matter cannot exist in stable orbits closer than 3 event horizons (i.e., 3 Schwarzschild radii) from the hole's center. Write down the expression for the radius of this innermost stable orbit.

2) Assume matter that eventually hits an accretion disk around a black hole starts out infinitely far away, with no velocity. It gains speed as gravity pulls it toward the hole. Using energy conservation, write down an expression for the kinetic energy of a mass m at radius r from the black hole (mass $M_{\rm BH}$). [You may neglect relativistic effects.]

3) When the mass makes its final approach toward the black hole and collides with other matter in the accretion disk, typically half of its kinetic energy gets converted into gas thermal energy with $\frac{3}{2}kT$ per particle and is eventually radiated away into space, and half goes into the disk or black hole itself. How much energy will be radiated due to mass m, assuming that the energy is released when the mass hits the innermost stable radius?

4) Based on your answer to 3), what is the approximate energy generation efficiency of accretion? How does this compare with fusion?

Gravitational Lensing - 15 pts

The gravitational curvature that an object induces upon Spacetime increases with mass and is a manifestation of the strength of gravity in that area. Light travels along these curved paths, called geodesics.

5) Massive objects bend passing light rays toward them. Consider a light ray that originates from a distant galaxy. As it passes near a massive object along its way to an observer (like an astronomer on Earth), it is deflected from the "straight" path it would take in the absence of this massive object by a small angle, called the deflection angle. Sketch a diagram of this, including the massive object, the actual path of the light ray, and what its path would be in the absense of the massive object. Label the deflection angle, as well as

the distance between the light ray and the massive object at the point of deflection (called the impact parameter b). You may assume that all the deflection occurs instantaneously.

6) Use dimensional analysis to determine the deflection angle α. That is, construct an expression that contains the relevant physical quantities (the mass of the deflecting body, for instance), along with the physical constants G and c, that has the units of radians, i.e. is dimensionless. [Dimensional analysis is useful in getting a rough idea of a number, but it doesn't give you the numerical factors that a real derivation (in this case, using general relativity) does. In this case, multiply your result by 4 to get the real deflection angle.]

7) Estimate roughly the deflection angle that a light ray experiences when passing (a) a star like the sun at its edge, (b) a galaxy like the Milky Way ($M \approx 6 \times 10^{11} M_{\odot}$, including dark matter) at 40 kpc, (c) a massive cluster of galaxies ($M \approx 10^{15} M_{\odot}$) at 3 Mpc. [Eddington observed the deflection of starlight by the sun during a solar eclipse in 1919. His observations were in agreement with Einstein's predictions, providing an important early test of general relativity.]

Crab Measurements - 12 pts

The Crab Nebula, M1, is the best known and best studied supernova remnant. The nebula is full of small knots and filaments, especially in its outer parts.

An image of the Crab taken in 2005 shows that two knots on diametrically opposite sides of the nebula are separated by 252 arcseconds. The same knots appear on images dug out of archives: in 1989, they were 249 arcseconds apart, in 1970, 243 arcseconds, and on the oldest images available, from 1939, they were 234 arcseconds from each other. Spectra of the nebula taken in 2004 show that the gas on the front side of the nebula is approaching us at 1260 kilometers per second.

8) From these data, estimate the distance to the Crab Nebula assuming a spherically symmetric expansion at constant velocity.

9) Estimate the physical diameter of the nebula in 2005 assuming its diameter is 252 arcseconds.

10) Estimate the age of the nebula (ignore the light-travel-time delay! Astronomers are almost always interested in when things happen as seen from Earth). Compare the age to the historical date of 1054 AD for the Crab supernova.

Fig. 1.— The Crab Nebula and its pulsar.