1/28/20

<u>Cosmological</u> <u>Principle</u>: The Universe is <u>homogeneous</u> and isotropic on large (~hundreds of Mpc) scales

[Aside: We will need to invoke Birkhoff's thm: "The metric inside a spherical cavity at the center of a spherically symmetric system is equivalent to a Minkowski flat-space metric – more later]

Redshift in Expanding Universe

Galaxy separations @ time t $\quad r_{ij}(t)$ in an expanding universe

- In order to stay homogeneous, we must have

$$\frac{r_{ij}(t)}{r_{12}(t)} = \text{constant}$$

$\uparrow$ some reference separation that expands w/everything else

Re-write as:

$r_{ij}(t) = a(t)\, r_{ij}(t_0)$, where $a(t)$ is some function

$\boxed{t_0 = \text{present time}}$

$\boxed{a(t_0) \equiv 1 \quad (\text{by convention})}$

$$\Rightarrow \quad v_{ij}(t) = \dot{r}_{ij}(t) = \dot{a}(t)\, r_{ij}(t_0) = \left(\frac{\dot{a}}{a}\right) r_{ij}(t) = \boxed{H(t)\, r_{ij}(t)}$$

where $\boxed{H(t) = \frac{\dot{a}(t)}{a(t)}}$ Hubble Parameter $\quad [v_0 = H_0 d]$

$$\frac{1}{a}\frac{da}{dt} = H$$

Thus, to maintain homogeneity + isotropy, $v \propto$ separation.
[All observers will observe the same effect]

1/28/20 (Sp. Rel) Doppler Shift emitted at one point and absorbed (or observed) at another:

$$\frac{\lambda_{obs}}{\lambda_{em}} = \left(\frac{c+v}{c-v}\right)^{1/2} \equiv (1+z)$$

or, $z \equiv \frac{\lambda_{obs}-\lambda_{em}}{\lambda_{em}} = \frac{\nu_{em}}{\nu_{obs}} - 1$

$\boxed{\lambda_{obs} = (1+z)\lambda_{em}}$, $z \equiv$ Redshift $\boxed{\nu_{obs} = \frac{\nu_{em}}{(1+z)}}$

Now, in time dt, photon travels a distance cdt, and is passing observers whose velocities differ from its point of emission by:

$\frac{d\lambda}{da} = \frac{a}{\lambda}$

① $dv = H(t) \cdot cdt$ → ② $\frac{dv}{c} = \boxed{\frac{da}{a} = \frac{d\lambda}{\lambda}}$ ⇒ $\lambda(a) = a\lambda_{obs}$

③ $\frac{dv}{c} = \frac{d\lambda}{\lambda} = \frac{da}{a} \Rightarrow \lambda_{obs} = C\lambda_{em}$

④ $\Rightarrow \lambda(a) = a\lambda_{obs}$, or

⇒ ⑤ $\boxed{(1+z) = \frac{a(t_0)}{a(t)} = \frac{1}{a(t)}}$ (since $a(t_0) \equiv 1$)

Thus, redshift indicates the relative scale factor (a) between emission and reception of a photon $\boxed{a = \frac{1}{(1+z)}}$

Note: "t" and "t_0" are consistent throughout universe, defined such that clocks were synchronized when matter was "on top of each other", or when $\rho(t) = \rho_0$ and one uses a reference frame in which universe appears isotropic and homogeneous ⇒ "Fundamental Observer"

[are we a F.O.?]

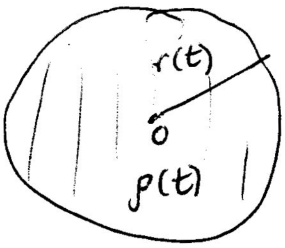
1/28/20

Dynamics of Expanding Universe

I. Newtonian derivation

$$r(t) = a(t)\, r(t_0) = a(t)\, r_c$$

co-moving coordinate – remains constant at all t



$\rho(t)$ = density inside $r(t)$

$$= \rho_0\, a^{-3}(t) = \rho_0 (1+z)^3$$

"Birkhoff's Thm" allows us to ignore gravity outside the volume

$$M(r_0) = \frac{4\pi}{3}\rho_0 r_0^3 = \frac{4\pi}{3}\rho(t)\, a^3(t)\, r_0^3 = M(r)$$ [Same mass, diff. vol.]

Gravitational acceleration

$$\ddot{r}(t) = \frac{d^2 r(t)}{dt^2} = -\frac{GM(r)}{r^2}$$

$$\dot{r} = \dot{a}(t)\, r_0, \quad \text{so} \quad \ddot{a} = \frac{\ddot{r}(t)}{r_0}$$

$$\Rightarrow \boxed{\ddot{a}(t) = -\frac{4\pi G}{3}\frac{\rho_0}{a^2(t)} = -\frac{4\pi G}{3}\rho(t)\, a(t)}$$

!! multiply both sides by $2\dot{a}$

$$\Rightarrow 2\dot{a}\ddot{a} = -\frac{4\pi G}{3}\rho a \cdot 2\dot{a} =$$

$$\frac{d(\dot{a}^2)}{dt} = 2\dot{a}\ddot{a} \quad \& \quad \rho a \dot{a} = \frac{\rho_0 \dot{a}}{a^2} = \rho_0 \frac{d(-1/a)}{dt}$$

$$\Rightarrow \frac{d(\dot{a}^2)}{dt} = -\frac{8\pi G}{3}\rho_0 \frac{d\left(-\frac{1}{a}\right)}{dt}$$

$\Rightarrow$ integrate over time,

$$\Rightarrow \dot{a}^2 - \frac{8\pi G}{3}\rho_0\left(\frac{1}{a}\right) = \text{constant} = -Kc^2$$

$$\rho_0 \cdot \left(\frac{1}{a}\right) = \rho a^2$$

$$\text{ⓐ} \quad \Rightarrow \boxed{\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - Kc^2}$$

$r = ar_0 \quad \dot{r} = r_0\dot{a}$

$\dot{a}^2 = \frac{\dot{r}^2}{r_0^2} \quad a^2 = \frac{r^2}{r_0^2}$

or,
$$\frac{\dot{r}^2}{r_0^2} = \frac{8\pi G}{3}\rho\frac{r^2}{r_0^2} - Kc^2$$

$m = \frac{4\pi}{3}r^3\rho \; ; \; \rho = \frac{M}{\frac{4\pi}{3}r^3}$

can be re-written,

$$\boxed{\frac{\dot{r}^2}{2} - \frac{GM}{r} = -\frac{Kc^2 r_0^2}{2}}$$

$-\frac{Kc^2}{2}r_0^2$ is total energy per unit mass

Kinetic + potential = total $E = -\frac{Kc^2 r_0^2}{2}$ = const.

$$\Rightarrow K = \frac{-2E}{c^2 r_0^2}$$

Note that if $K<0$, total E is always positive

$\Rightarrow$ if $\dot{a}$ = positive now, it always was and always will be

if $K=0$,

RHS of eqn ⓐ above is always positive, but asymptotically $\dot{a}(t) \to 0$ as $t \to \infty$

$$\rightarrow \frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}G\rho - \frac{Kc^2}{a^2}$$

$K>0$, then RHS of ⓐ, there will be an a_{max} where expansion will reverse

For perfect balance between kinetic + potential ($K=0$)

$$\dot{a}^2 = \frac{8\pi}{3}G\rho a^2 \; ; \; \frac{\dot{a}^2}{a^2} = H^2 = \frac{8\pi}{3}G\rho$$

$\Rightarrow \rho = \rho_{crit}$ at present (t_0) becomes $H_0^2 = \frac{8\pi}{3}G\rho_0$

$\Rightarrow 1.88 \times 10^{-29} h^2 \, g/cm^3 \qquad (100h)^2 =$

$\approx 0.94 \times 10^{-29}$ for $h = 0.7$

Define $\Omega_0 \equiv \frac{\rho_0}{\rho_{crit,0}} \Rightarrow \begin{cases} \Omega_0 > 1 & \text{max exp} \\ \Omega_0 = 1 & \text{expand forever} \\ \Omega_0 < 1 & \text{expand forever} \end{cases}$

Relativistic Additions (adding Pressure)

For co-moving, expanding sphere, balance work done by internal pressure in expanding by δr:

$$\underbrace{\delta \left[\tfrac{4}{3}\rho r^3 c^2\right]}_{\text{total mass-energy density}} + \underbrace{4\pi r^2 P \delta r}_{\text{work done by internal pressure}} = 0$$

change in internal energy

Note: $\frac{d}{dr}(\rho r^3) = -\frac{3P}{c^2} r^2$; $r = r_0 a(t)$

$$\Rightarrow \frac{d}{da}(\rho a^3) = -\frac{3P}{c^2} a^2$$

Note $\frac{d}{da} = \frac{d}{dt}\left(\frac{da}{dt}\right)^{-1} = \frac{d}{dt}\left(\frac{1}{\dot{a}}\right)$

or $\frac{1}{\dot{a}} \frac{d(\rho a^3)}{dt} = -\frac{3P}{c^2} a^2$

$$\frac{1}{\dot{a}}\left[3\rho \cancel{a^2} \dot{a} + a^{\cancel{3}} \dot{\rho}\right] = \frac{-3P}{c^2} \cancel{a^2}$$

$$3\rho + \frac{a}{\dot{a}}\dot{\rho} = -\frac{3P}{c^2} \Rightarrow \boxed{\dot{\rho} + 3\left(\rho + P/c^2\right)\frac{\dot{a}}{a} = 0}$$

$$\dot{\rho} + 3\left(\rho + P/c^2\right)\left(\frac{\dot{a}}{a}\right)$$

Energy Cons. Eqn

Geometry

Plane: $\alpha + \beta + \gamma = \pi$ (for a triangle)

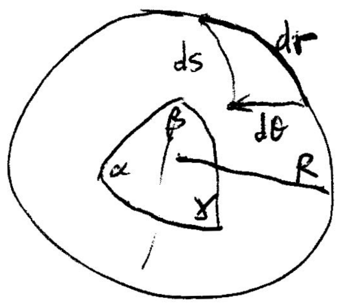
Cartesian coords: (x, y)

distances: $ds^2 = dx^2 + dy^2$

Polar coords: (r, θ)

$$ds^2 = dr^2 + r^2 d\theta^2$$

Surface of a Sphere:



$$\alpha + \beta + \gamma = \pi + \frac{A}{R^2}$$

area of triangle

radius of sphere

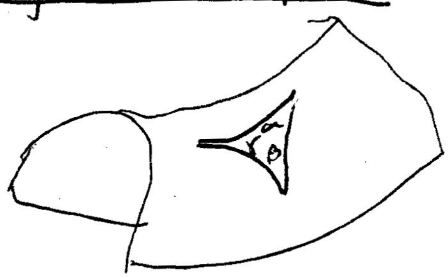
$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2$$

$r =$ distance from "north pole"

maximum separation $= \pi R$

Sphere has "constant positive curvature"

Negatively Curved Space (e.g., "Saddle")



$$\alpha + \beta + \gamma = \pi - \frac{A}{R^2}$$

on surface, $ds^2 = dr^2 + R^2 \sinh\left(\frac{r}{R}\right) d\theta^2$

A curved space that is uniformly curved can be specified by R, K

R ↑ radius of curvature; K ↑ $+1, -1, 0$

Now, extend the 2-d surface to 3-d space:

$K=0$: $ds^2 = dx^2 + dy^2 + dz^2$ (Cartesian)

$ds^2 = dr^2 + r^2\left[d\theta^2 + \sin^2\theta \, d\phi^2\right]$ (Spherical coords)

[infinite volume]

$K=+1$ $\quad ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\left[d\theta^2 + \sin^2\theta \, d\phi^2\right]$

Note that positive curvature ⇒ space is finite

[e.g., $C = 2\pi R$ would circumnavigate the whole space]

$K=-1$ $\quad ds^2 = dr^2 + R^2 \sinh^2\left(\frac{r}{R}\right)\left[d\theta^2 + \sin^2\theta \, d\phi^2\right]$

Shorthand: ① $\boxed{ds^2 = dr^2 + S_K(r)^2 \, d\Omega^2}$

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

$$S_K(r) = \begin{cases} r & (K=0) \\ R\sin\left(\frac{r}{R}\right) & K=+1 \\ R\sinh\left(\frac{r}{R}\right) & K=-1 \end{cases}$$

N.B.: all 3 have $S_K \approx r$ for $r \ll R$

$S_K \to \infty$ for $K=0$, $K=-1$ at large r

$S_K \to$ max at $\frac{r}{R} = \frac{\pi}{2}$ for $K=+1$

If we set $x \equiv S_K(r)$,

② $\boxed{ds^2 = \frac{dx^2}{1 - Kx^2/R^2} + x^2 \, d\Omega^2}$ equivalent to ① above

Now, special relativity introduces t as a fourth dimension, where the space-time separation of 2 events is

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2$$ "Minkowski metric" (flat space)

Photons travel on "null geodesics", i.e.

$$ds^2 = 0 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2$$

For a photon moving on a radial path toward/away from origin, $\left.\begin{matrix} d\theta \\ d\phi \end{matrix}\right\}$ are zero (constant θ, ϕ)

$$\Rightarrow \quad c^2 dt^2 = dr^2$$

[Ended here, 1/28/20] and $\frac{dr}{dt} = \pm c$

Now, suppose space is <u>not</u> flat, but remains homogeneous + isotropic

$\Rightarrow$ Robertson-Walker Metric

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{d\chi^2}{1 - \frac{K\chi^2}{R_0^2}} + \chi^2 d\Omega^2 \right] \quad \underline{\underline{OR}}$$

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_\kappa(r)^2 d\Omega^2 \right]$$

(a) - t is time seen by an observer who sees space expanding around her, and (b) (r, θ, ϕ) or (χ, θ, ϕ) are the co-moving coordinates

(c) R_0 - radius of curvature at present moment

(d) $K = 0, +1, -1$

Distances in Expanding Universe

① Proper distance $d_p(t)$ between 2 points is the length of the spatial geodesic between them when $a(t)$ is fixed.

$$ds^2 = a(t)^2\left[dr^2 + S_k(r)^2 \, d\Omega^2\right]$$

or, $ds = a(t)\, dr$

So, $d_p(t) = a(t)\int_0^r dr = \boxed{a(t)\, r}$

If we fix the co-moving coordinate r,

$$\dot{d}_p = \dot{a}\, r = \frac{\dot{a}}{a} d_p$$

i.e., $v_p(t_0) = H_0 d_p(t_0)$; $H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_0}$

Einsteins' field equations give a mathematical relation between space-time metric at a point and the energy and pressure at that point.

We have already derived the F.E. in a Newtonian universe,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2E}{r_0^2}\frac{1}{a(t)^2}$$

The correct form is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\frac{G}{c^2}\varepsilon(t) - \frac{Kc^2}{R_0^2}\frac{1}{a(t)^2}$$

Note $\rho \rightarrow \frac{\varepsilon}{c^2}$ recall $E = (m^2c^4 + p^2c^2)^{1/2}$

If $v \ll c$, $p \approx mv$, $E = mc^2(1 + v^2/c^2) \approx mc^2 + \frac{1}{2}mv^2$

But for photons,

$$E_{rel} = pc = h\nu \quad , \text{ also contribute to } \mathcal{E}$$

Note that in F.E., $\dfrac{2E}{r_0^2} = -\dfrac{Kc^2}{R_0^2}$

Thus, $$H(t)^2 = \frac{8\pi G}{3c^2}\,\mathcal{E}(t) - \frac{Kc^2}{R_0^2\,a(t)^2}$$

$$H_0^2 = \frac{8\pi G}{3c^2}\,\mathcal{E}_0 - \frac{Kc^2}{R_0^2}$$

and the critical energy density is $\mathcal{E}_c(t) = \dfrac{3c^2}{8\pi G} H(t)^2$

critical mass density $\rho_{c,0} = \dfrac{\mathcal{E}_{c,0}}{c^2}$

$$\Omega(t) \equiv \frac{\mathcal{E}(t)}{\mathcal{E}_c(t)}$$

Note, then, that

$$1 - \Omega(t) = \frac{-Kc^2}{R_0^2\,a(t)^2 H(t)^2}$$

note this cannot change sign

$\Rightarrow$ if $\Omega(t) > 1$ at any time, it is true for all time

< 1 " "

$= 1$

$$\frac{K}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1)$$

$\Longrightarrow$ if you know Ω_0, $\Rightarrow$ sign of K

if you know $\frac{c}{H_0}$, $\Rightarrow$ can compute R_0

We can re-do the 1st Law of Thermodynamics now:

$$dQ = dE + PdV$$

dQ: heat flow into or out of region (zero for expanding universe)

dE: change in internal energy

dV: change in volume

$$\Rightarrow \text{(a)}\ \dot{E} + P\dot{V} = 0$$

$$V(t) = \frac{4\pi}{3} r_s^3 a(t)^3$$

$$\text{(b)}\ \dot{V}(t) = \frac{4\pi}{3} r_s^3 (3a^2\dot{a}) = V\left(3\frac{\dot{a}}{a}\right)$$

and, $\text{(c)}\ E(t) = V(t)\,\varepsilon(t)$

$$\Rightarrow \dot{E} = V\dot{\varepsilon} + \dot{V}\varepsilon = V\left(\dot{\varepsilon} + 3\frac{\dot{a}}{a}\varepsilon\right)$$

$$\Rightarrow \text{(a), (b), (c)} \Rightarrow V\left(\dot{\varepsilon} + \frac{\dot{a}}{a}\varepsilon + 3\frac{\dot{a}}{a}P\right) = 0$$

$$\Rightarrow \text{(d)}\ \boxed{\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0} \quad \text{"Fluid Equation"}$$

F.E. $\dot{a}^2 = \frac{8\pi G}{3c^2}\varepsilon a^2 - \frac{Kc^2}{R_0^2}$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\left(\dot{\varepsilon}a^2 + 2\varepsilon a\dot{a}\right)$$

divide by $2\dot{a}a \Rightarrow$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(\dot{\varepsilon}\frac{a}{\dot{a}} + 2\varepsilon\right)$$

$$\text{(d)} \Rightarrow \quad \dot{\varepsilon}\frac{a}{\dot{a}} = -3(\varepsilon + P) \Rightarrow \boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)}$$

Equations of State

This is the last piece of information we need to fully specify $a(t)$, $\epsilon(t)$, $P(t)$

$$P = P(\varepsilon) = \underset{\substack{\uparrow \\ \text{dimensionless}}}{w}\epsilon$$

(1) Non-rel. gas

For a non-relativistic gas, the ideal gas law says

$$P = \frac{\rho}{\eta} kT \qquad P = \frac{kT}{\eta c^2}\varepsilon$$

"m" — but ε: almost entirely the mass of gas particles

$$E = mc^2(1 + \underbrace{v^2/c^2}_{\ll 1}) \simeq mc^2, \text{ so}$$

Also, kinetic and thermal energy / particle are equal,

$$3kT = \eta\langle v^2\rangle \qquad \frac{3}{2}kT = \frac{1}{2}\eta\langle v^2\rangle$$

$$\Rightarrow P = w\varepsilon_{\text{nonrel}} \; ; \; w \simeq \frac{\langle v^2\rangle}{3c^2} \ll 1$$

So, in general, P can be neglected for nonrel. gas

(2) Photons or Relativistic gas

$$P_{\text{rel}} = \frac{1}{3}\varepsilon_{\text{rel}} \; ; \; w = \frac{1}{3} \qquad \text{[Why?]}$$

and, the "sound speed" in a relativistic fluid $c_s^2 = c^2\left(\frac{dP}{d\varepsilon}\right) \Rightarrow \boxed{c_s = \sqrt{w}\, c}$

We will call substances w/ $w = 0$ "matter" and substances w/ $w = \frac{1}{3}$ is radiation.

Note that the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)$$

flips sign to $\overset{\nearrow \text{acceleration}}{}$ if $\varepsilon + 3P \to < 0$

i.e., $0 > \varepsilon + 3P$

$\Rightarrow w \leq -\frac{1}{3}$

$$\boxed{\begin{array}{c}\text{A cosmological constant has } w = -1 \\ P = -\varepsilon\end{array}}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda}{3}$$ Friedmann

$\Lambda = 4\pi G\rho \Rightarrow$ static sol'n

and acceleration eqn becomes

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) + \frac{\Lambda}{3}}$$

N.B adding the Λ term is equivalent to adding a new component with

$$\varepsilon_\Lambda = c^2/8\pi G \, \Lambda$$

The fluid eqn, $\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$

$\Rightarrow$ if Λ is constant, then

$$P_\Lambda = -\varepsilon_\Lambda = -\frac{c^2}{8\pi G}\Lambda$$

|| Thus, cosm. constant has constant density ε_Λ and constant pressure, $P_\Lambda = -\varepsilon_\Lambda$

We can split the mass-energy density into components contributed by different constituents:

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = \frac{8\pi G}{3}\left[\rho_m + \rho_{rad} + \rho_\Lambda\right] - \frac{Kc^2}{a^2}$$

$\rho_\Lambda \to$ constant

$\rho_m \propto (1+z)^3 \propto a^{-3}$

$\rho_{rad} \propto (1+z)^4 \propto a^{-4}$ because $\rho_{rad} = a_{SB} T^4$

$$\Rightarrow H^2(t) = \frac{8\pi G}{3}\left[\rho_{0,m}(1+z)^3 + \rho_{0,r}(1+z)^4 + \rho_{\Lambda,0}\right] - \frac{Kc^2}{a^2}$$

or, $$H^2(t) = H_0^2\left[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0} + \left(\frac{1-\Omega_{m,0}-\Omega_{\Lambda,0}}{a^2}\right)\right]$$

curvature term

If the curvature $=0$,

$$H^2(t) = H_0^2\left[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}\right]$$
$$= H_0^2 E^2(z)$$

where $\boxed{E(z) = H_0\left[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}\right]^{1/2}}$

We will use "r" to represent the co-moving coordinate distance, so

$$ds^2 = a(t)\,dr \quad ; \quad \int ds = \int_0^r a(t)\,dr = a(t)\,r$$

"Proper distance"

$$a(t)\,dr = c\,dt$$

$$\Rightarrow D_c(t) = \int_{t=t_{em}}^{t_0} \frac{c\,dt}{a(t)}$$

So, a removes the expansion of the universe

F.E. says: $\dot{a}^2 = H^2 a^2$; $\frac{da}{dt} = H(t)\,a(t)$; $dt = \frac{da}{da/dt} = \frac{da}{\dot{a}}$

$$\Rightarrow \frac{da}{\dot{a}} = \frac{da}{aH} \Rightarrow \int_{t_{em}}^{t_0} \frac{c\,dt}{a} = \int_{a_{em}}^{1} \frac{c\,da'}{a'^2 H}$$

Now, change variables:

$$a = (1+z)^{-1}$$

$$da = -(1+z)^{-2} dz = -a^2 dz$$

We had $\int_{t_{em}}^{t_0} \frac{c\,dt}{a} = \int_{a_{em}}^{1} \frac{c\,da'}{\dot{a}^2 H} = \int_0^z \frac{c}{H(z)} dz = \int_0^z \frac{c}{H_0} \frac{dz'}{E(z')}$

$$\left(H_0 E(z) = H(z) \right)$$

$$\Longrightarrow \boxed{D_c = \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')}}$$ "Co-moving distance to redshift z"

$$E(z) = \left[\Omega_m (1+z)^3 + \Omega_r (1+z)^4 + \Omega_\Lambda + \underbrace{\Omega_K}_{= \frac{1-\Omega_m-\Omega_\Lambda}{a^2}} \right]^{1/2}$$

Special case: $K=0$ "Einstein-de Sitter" Universe

$$\Omega_m = 1, \quad \Omega_\Lambda = 0$$

$$E(z) = \left[(1+z)^3 \right]^{1/2} = (1+z)^{3/2}$$

$$\Longrightarrow D_c(z) = \frac{c}{H_0} \int_0^{z'} \frac{dz'}{(1+z)^{+3/2}} = \frac{2c}{H_0} \left[1 - (1+z)^{-1/2} \right]$$

Note: $\frac{c}{H_0} = 4300 \text{ Mpc} \Longrightarrow D_c = 8600 \text{ Mpc} \left[1 - (1+z)^{-1/2} \right]$

e.g., co-moving distance to $z=2$

$$= \frac{2c}{H_0} \left[1 - 3^{-1/2} \right] \approx 0.84 \frac{c}{H_0}$$

"Proper Distance"

With $a(t)$ fixed, physical distance between 2 points

$$D_p = a(t)\int_0^r dr = a(t)\cdot r = a(t)\,D_c = \frac{D_c}{(1+z)}$$

Also, $\dot{D}_p = \dot{a}\,r = \frac{\dot{a}}{a} D_p = H\,D_p$

$$\Rightarrow V_p(t_0) = H_0 D_p(t_0) \quad \text{where } H_0 = \left(\frac{\dot{a}}{a}\right)_{t=0}$$

Lookback Time

$$D_p(t) = a(t)\int_0^r dr = \int c\,dt = a(t)\cdot D_c$$

$$\int_{t_{em}}^{t_0} dt = t_{LB} = \frac{D_p(z)}{c} = \frac{1}{H_0}\int \frac{da'}{a' E(a')} \qquad H^2 = H_0^2 E^2$$

$$a\,E(a) = \left[a^2\Omega_\Lambda + \frac{\Omega_m}{a} + \frac{\Omega_r}{a^2} + (1-\Omega_0)\right]^{1/2}$$

For E-de S: $\Omega_m = 1$, $\Omega_\Lambda = \Omega_k = 0$

$$\Rightarrow a\,E(a) = a^{-1/2}\Omega_m = a^{-1/2}$$

$$\Rightarrow t_{LB}(a) = \frac{1}{H_0}\int_a^1 a^{1/2}\,da = \frac{2}{3H_0}\left[1 - a^{3/2}\right]$$

$$\Rightarrow \boxed{t(z) = \frac{2}{3H_0}\left[1 - (1+z)^{-3/2}\right]}$$

$$\Rightarrow \boxed{t(z=\infty) = \frac{2}{3H_0}}$$