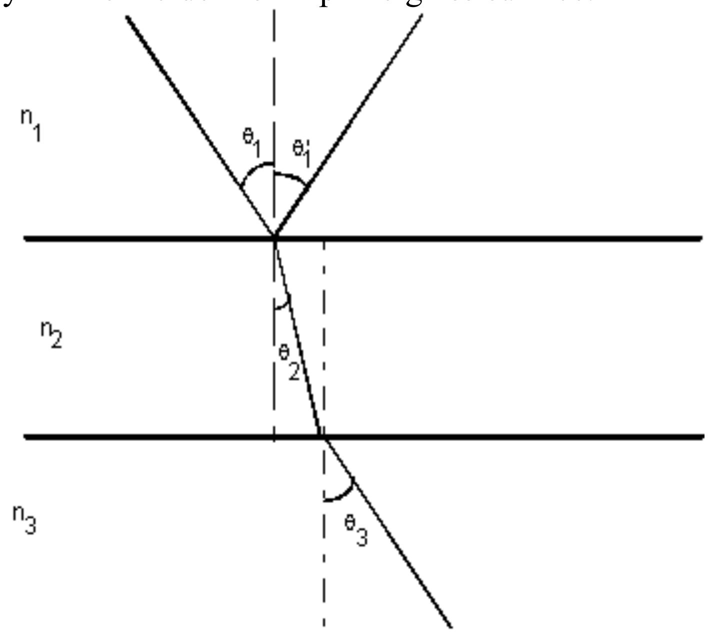
# Geometrical Optics

We have seen that light is an electromagnetic wave. We have also seen that waves can be reflected and refracted by various surfaces placed in the path of the wave. How can we apply this to light? To a high degree of accuracy, we can imagine that light waves move in a straight line, provided that we do not place any obstacle or aperture in the path of the wave that has the same magnitude of dimensions as the wavelength of the wave. If we also disregard the effect of edges on the light wave, we have the special case known as **geometrical optics**. In this case, we view the light wave as a **ray** which is always normal to the surface of the wave front.

Consider a light ray that is incident on a plane glass surface.




We see that the beam is partially reflected and partially transmitted. The transmitted beam is bent both upon entering the surface and upon exiting it. Let $q_1$ be the angle the incident beam makes with the normal to the surface of the glass. Also, let $q_1'$ be the angle the reflected beam makes with the normal, $q_2$ be the angle of the beam in the glass surface and $q_3$ be the angle of the beam that exits the glass surface. By experiment, we find that the reflected and refracted rays lie in the plane formed by the incident ray and the normal to the surface at the point of incidence. This plane is called the **plane of incidence**.

## Laws of Reflection and Refraction

We also find that, for the reflected ray,

$$q_1 = q_1'. \tag{173}$$

This is called the **law of reflection**. For the transmitted ray, we have that

$$n_1 \sin q_1 = n_2 \sin q_2 \tag{174}$$

where $n_i$ is the **index of refraction** of the $i^{th}$ medium. (174) is called the **law of refraction**, or **Snell's law**. The index of refraction can be related to the electric permittivity and the magnetic permeability of the material (the magnetic permeability of a material acts on the magnetic field in a manner similar to the action of the permittivity on the electric field). Specifically, we have that

$$n^2 = \frac{\varepsilon\mu}{\varepsilon_0\mu_0} \tag{175}$$

Recall that the electric permittivity is related to the dielectric constant by $e = Ke_0$. Notice that in a vacuum, $e = e_0$ and $m = m_0$, so $n = 1$ in a vacuum.

Example:

A light ray is passing from air into water. If the index of refraction for air is $n = 1$ and the incident angle is 30°, what is the angle of the refracted beam?

The index of refraction for water is $n = 1.33$. By Snell's law, the angle of the refracted beam is

$$\theta_2 = \sin^{-1}\frac{n_1 \sin\theta_1}{n_2} = \sin^{-1}\frac{\sin(30°)}{1.33} = 22.1°$$

Example:

For the beam given in the above example, what is the angle the beam makes with the normal as it exits the water?

In this case, the roles of $n_1$ and $n_2$ are interchanged. Then Snell's law yields

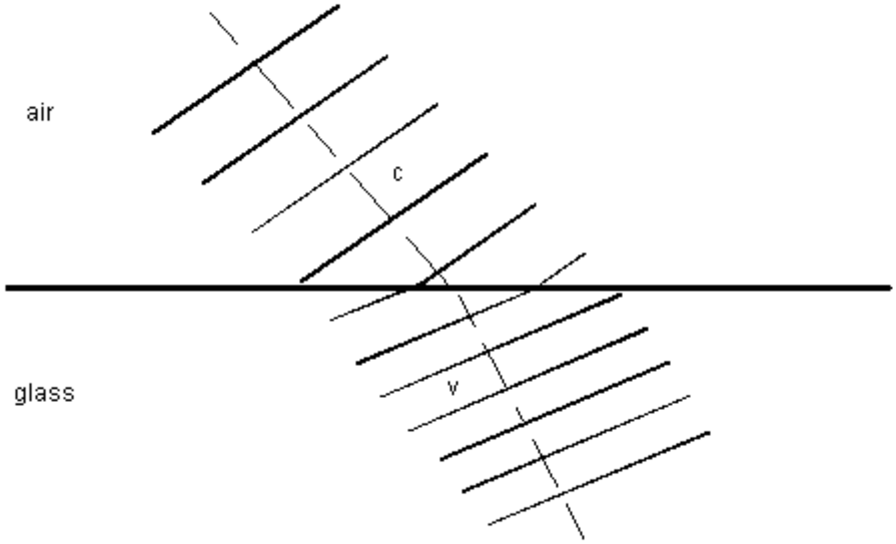
$$\theta_1 = \sin^{-1}\frac{n_2 \sin\theta_2}{n_1} = \sin^{-1}\frac{(1.33)\sin(22.1°)}{1} = 30.0°$$

These two examples illustrate an important fact. **In going from a medium with a small index of refraction to one with a large index of refraction, the ray will bend towards the normal; in going from a medium with a large index of refraction to one with a small index of refraction, the ray will bend away from the normal.**

# Huygen's Principle

An important principle that takes account of the wave-like nature of light is **Huygen's principle**. It states that **all points on a wave front can be considered as point sources for the production of spherical secondary wavelets. After a time *t* the new position of the wave front will be the surface of tangency to these secondary wavelets**. This principle allows us to tell where a wave front will be at any time in the future if we know its present position. We can use Huygen's principle to find the speed of light in a medium with an index of refraction greater than 1.

Consider the passage of a light wave from the air into glass.




If the light has a velocity v in the glass, then in a time *t*, the wave will travel a distance $x_1/c$ in air and $x_2/v$ in the glass. Thus, we have that

$$\frac{x_1}{c} = \frac{x_2}{v}$$

But, we can replace $x_1$ and $x_2$ by $\mathrm{sinq}_1$ and $\mathrm{sinq}_2$ since $x_i = l\,\mathrm{sinq}_i$. Multiplying through by *c*, we find that

$$\sin\,\theta_1 = \frac{c}{v}\sin\,\theta_2$$

which is just Snell's law if we make the association

$$n = \frac{c}{v} \qquad (176)$$

Since $n$ ³ 1 always, we see that $v$ £ $c$ in materials. This is consistent with the postulates of special relativity and with the definition of n given in (175).

Notice that the wavelength of light will also change in a material. This follows from the relation

$$lf = c$$

If we replace *c* with *c*/*n*, then the wavelength becomes

$$\lambda = \frac{c}{nf} = \frac{\lambda_0}{n} \tag{177}$$

where $l_0$ is the wavelength in vacuum.

# Total Internal Reflection

Recall that we saw that, when we go from a dense medium to a thin one, the refracted ray is bent away from the normal. This brings up the question of whether it is possible to bend the ray so that it does not exit from the medium. The answer to this is yes, and we can find the **critical angle**, $q_c$, from Snell's law. Let $n_2 < n_1$. Then if we set $q_1 = q_c$ and $q_2 = 90°$ we get

$$\begin{aligned} \theta_c &= \sin^{-1} \frac{n_2 \sin(90°)}{n_1} \\ &= \sin^{-1} \frac{n_2}{n_1} \end{aligned} \tag{178}$$

When we have $q > q_c$, the refracted ray cannot exit the medium and we have **total internal reflection**.

<u>Example</u>:
What is the critical angle for total internal reflection for a glass of water?

The index of refraction for water is 1.33, while the index of refraction for glass is 1.5. If we are going from glass into water, we have that $n_1 = 1.5$ and $n_2 = 1.33$. Then the critical angle is

$$\theta_C = \sin^{-1}\frac{n_2}{n_1} = \sin^{-1}\frac{(1.33)}{(1.5)} = 62.5^\circ$$

If we are going from the water into glass, $n_1$ = 1.33 and $n_2$ = 1.5. Thus, $n_2 > n_1$, so the condition for internal reflection is not satisfied.

# Polarization of Light

So far, we have seen that when we go from one medium to another we get both reflection and refraction of the incident ray. The reflected ray is at the same angle as the incident ray and the refracted ray is at an angle proportional to the ratio of the indexes of refraction for the two mediums. We have also seen that the velocity and wavelength of the incident ray is changed as it enters a different medium. The last thing that we want to show is that the interface between two mediums can polarize a ray of light.

Consider a ray of light passing from air into glass. We can decompose this ray into components that are parallel and perpendicular to the air/glass interface. In unpolarized light these two components are equal in magnitude. However, if the angle of incidence is at a specific angle, known as the **polarization angle** $q_p$, then the reflection coefficient (the amount of light reflected) of the parallel component is zero. From experiment, we find that the polarization angle $q_p$ and the refraction angle $q_r$ are at right angles, i.e.

$$q_p + q_r = p/2.$$

Plugging this into Snell's law, we find that

$$n_1 \sin\theta_p = n_2 \sin\theta_r = n_2 \sin\left(\frac{\pi}{2} - \theta_p\right) = n_2 \cos\theta_p$$

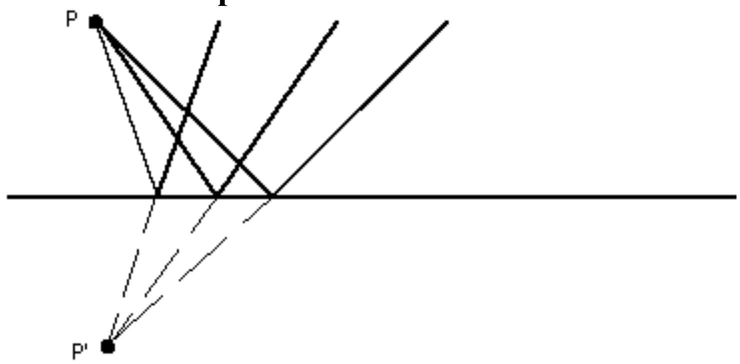
or

$$\theta_p = \tan^{-1}\frac{n_2}{n_1} \qquad (179)$$

This is known as **Brewster's law**. Note that, while we derived it for air and glass, the result holds in general.

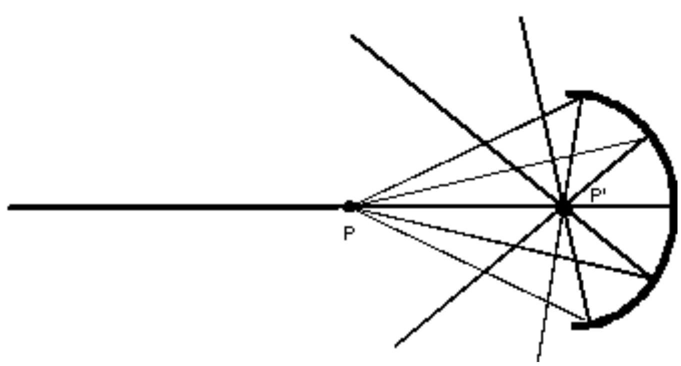
# Mirrors

Let us now apply these laws of reflection and refraction to the formation of images by mirrors and lenses. Consider first a **plane mirror**.




The figure shows a narrow bundle of light rays coming from a point source *P* reflected in a plane mirror. After reflection, the rays diverge exactly as if they came from a point *P*' behind the plane of the mirror. The point *P*' is called the **image** of the object *P*. When these rays enter the eye, they cannot be distinguished from rays diverging from a source *P*' with no mirror. The image is called a **virtual image** because the light does not actually emanate from the image but only appears to. Geometric construction using the law of reflection shows that the image lies on the line through the object perpendicular to the plane of the mirror and at a distance behind the plane equal to the distance from the plane to the object. If the source is an extended object, the size of the image formed is the same size as that of the object.

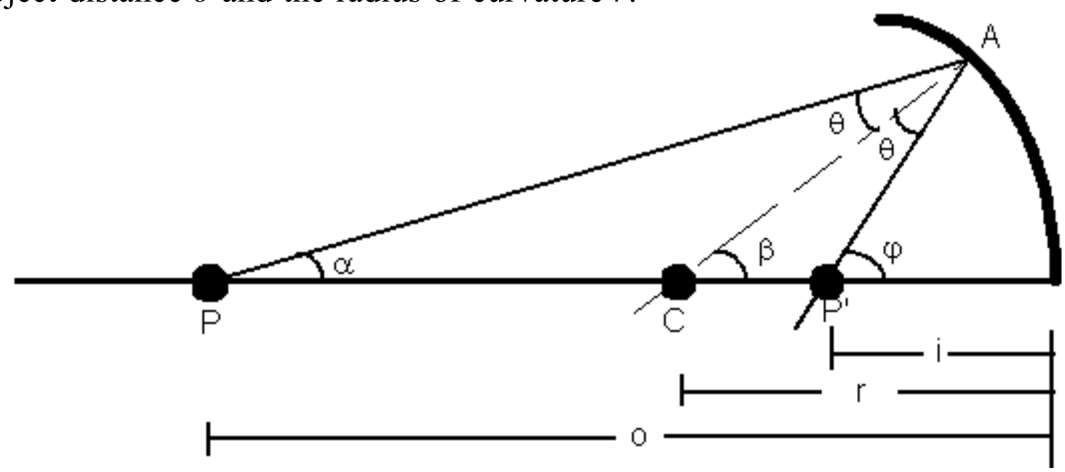
Let us now look at a **spherical mirror**.




This figure shows a bundle of rays from a point on the axis of a concave spherical mirror reflecting from the mirror and converging at point *P*'. The rays then diverge from this point just as if there were an object at that point. this image is called a **real image** because the light actually does emanate from the image point. It can be seen by an eye placed to the left of the image looking into the mirror. It could also be observed on a viewing screen placed at the image point. A virtual image cannot be observed on a screen at the image point because there is no light there. Despite this distinction between real and virtual images, the light rays diverging from a real image and those appearing to diverge from a virtual image are identical, so that no distinction is made by the eye when viewing either a real or virtual image.

# Focal Length

Using the law of reflection and elementary geometry, we can relate the image distance *i* to the object distance *o* and the radius of curvature *r*.




The angle b is an exterior angle to the triangle *PAC* and therefore b = a + q. Similarly, from triangle *PAP*', we have j = a + 2q. Eliminating q from these two equations gives 2b = j + a. When these angles are small, they are related to the image distance, the object distance and the radius of curvature by $\alpha \approx 1/o$, $\beta \approx 1/r$, and $\varphi \approx 1/i$. Thus the result is

$$\frac{1}{o}+\frac{1}{i}=\frac{2}{r} \qquad (180)$$

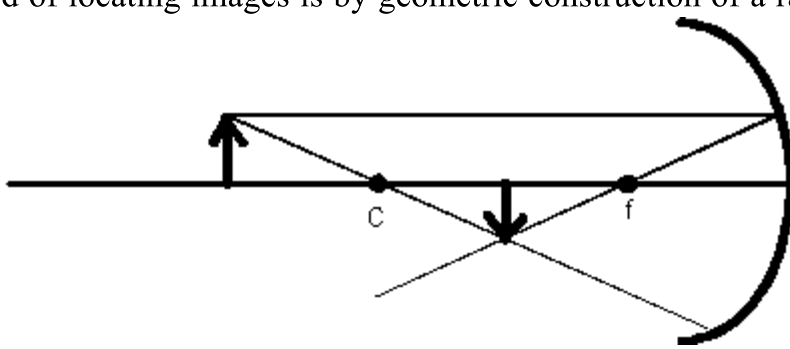
The derivation of this equation assumes that angles made by the incident and reflected rays with the axis are small. When the object distance is much greater that the radius of curvature of the mirror, the term 1/*o* in (180) can be neglected, resulting in $i = \frac{1}{2}r$ for the image distance. This distance is called the **focal length** *f* of the mirror. In terms of the focal length f the mirror equation (180) is

$$\frac{1}{o}+\frac{1}{i}=\frac{1}{f} \qquad (181)$$

The focal point is the point at which parallel rays corresponding to plane waves from infinity are focused. Note that we can also place a point source at the focal point and it will reflect rays which are parallel to the axis. This illustrates the property of waves called **reversibility**. If we reverse the direction of a reflected ray, the law of reflection assures us that the reflected ray will be along the original incoming ray but in the opposite direction. Reversibility holds also for refracted rays.

# Ray Diagrams

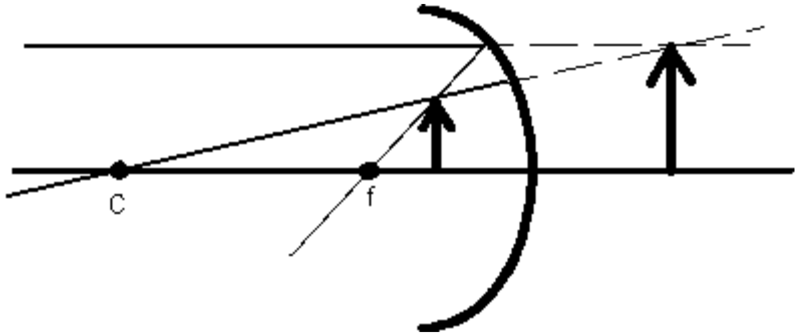
A useful method of locating images is by geometric construction of a ray diagram.




By a judicious choice of rays from the head of the object we can quickly locate the image. A ray from the head parallel to the axis is reflected through the focal point a distance $\frac{1}{2}r$ from the mirror. Another ray, through the center of curvature of the mirror, strikes the mirror perpendicular to the surface and is reflected back along its original path. The intersection of these two rays locates the image point of the head.

We see from the above figure that the image is inverted and is not the same size as the object. The magnification of the optical system (the spherical mirror in this case) is defined to be the ratio of the image size to the object size. Mathematically, this ratio is also the ratio of the distances *i* and *o*.

When the object is between the mirror and its focal point, the rays reflected from the mirror do not converge but appear to diverge from a point behind the mirror.



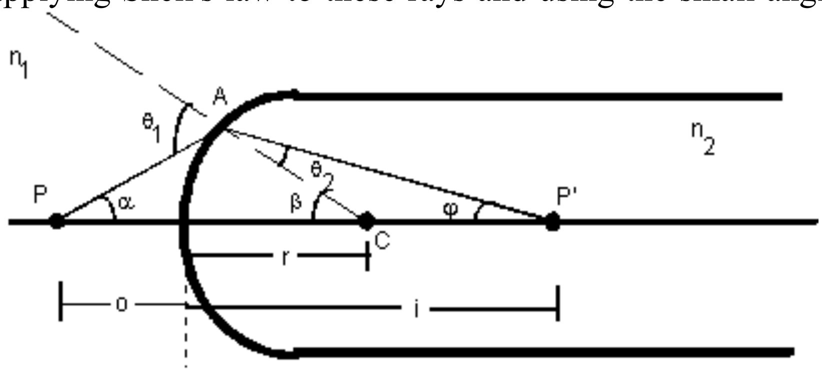

In this case the image is virtual and erect. If *o* is less that $\frac{1}{2}r$ in (180), the image distance *i* turns out to be negative. We can apply equations (180) and (181) to this case and to convex mirrors if we adopt a convenient sign convention. Whether the mirror is convex or concave, real images can be formed only on the same side of the mirror as the object and virtual images are formed on the opposite side, where there is no actual light rays. Distances to points on the real side are taken to be positive; distances to points on the virtual side are taken to be negative. Thus for a concave mirror, *o* and *r* are positive, and *i* is positive or negative depending on whether the image is real or virtual. For a convex mirror the center of curvature is on the virtual side, and so *r* is taken to be negative. The focal length is also negative. For either case, equation (180) gives the image distance *i* in terms of the object distance and radius of curvature. The lateral magnification of the image is given by

$$m = \frac{y'}{y} = -\frac{i}{o} \tag{182}$$

where *y*' and *y* are the heights of the image and the object respectively. A negative magnification, which occurs when both *o* and *i* are positive, indicates that the image is inverted.

# Image Formation via Refraction

Now let's consider the formation of an image by refraction from a spherical surface separating two media with indexes of refraction $n_1$ and $n_2$. We can derive an equation relating the image distance to the object distance, the radius of curvature, and the indexes of refraction by applying Snell's law to these rays and using the small angle approximation.




The angles $q_1$ and $q_2$ are related by Snell's law, which for small angles can be written $n_1\theta_1 = n_2\theta_2$. From the triangle *ACP*' we obtain $n_1 q_1 = n_2 b - n_2 j$. From the triangle *PAC* we obtain $q_1 = a + b$. Eliminating $q_1$ gives $n_1 a + n_2 j = (n_2 - n_1) b$. When these angles are small, they are related to the image distance, the object distance and the radius of curvature by $\alpha \approx 1/o$, $\beta \approx 1/r$, and $\varphi \approx 1/i$. Thus, the result is

$$\frac{n_1}{o} + \frac{n_2}{i} = \frac{n_2 - n_1}{r} \qquad (183)$$

We can use the same sign convention for this equation, but we must note that for refraction real images are formed to the right of the surface (if the object is to the left) and virtual images to the left. Thus *i* and *r* are taken to be positive if the image and center of curvature lie to the right of the surface.

# Lensmaker's Equation

The most important application of equation (183) is in the finding of the image formed by a lens. We do this by considering the refraction at each surface separately. We shall consider a lens of index of refraction $n_2$ embedded in a medium of index of refraction $n_1$. Let the radii of curvature of the surfaces of the lenses be $r_1$ and $r_2$. If an object is at a distance *o* from the first surface, application of (180) gives the distance of the image due to refraction at the first surface as

$$\frac{n_1}{o}+\frac{n_2}{i_1}=\frac{n_2-n_1}{r_1} \tag{184}$$

This image is usually not formed (unless the lens is extremely thick) because the light is again refracted at the second surface. If the thickness of the lens is *t*, the distance from the image point $i_1$ to the second surface is $t - i_1$. We can find the final image position due to both refractions by using this distance for the object distance for the second surface. It turns out that, for all possible values of the first image distance $i_1$, the image formed by refraction at the second surface is at a distance *i* from the second surface, where *i* is given by

$$\frac{n_2}{t-i_1}+\frac{n_1}{i}=\frac{n_1-n_2}{r_2} \tag{185}$$

For a general lens of thickness *t*, it is usually easier to find the distance $i_1$ numerically from (184) and use the result in (185) to find *i* than to eliminate $i_1$ from these two equations. However, in many cases, the thickness *t* is much smaller than any of the other distances involved. For such a **thin lens** we can neglect *t* in (184) and easily eliminate $i_1$ from these equations. Solving for $n_2/i_1$ in each equation, we obtain

$$\frac{1}{i}+\frac{1}{o}=\frac{n_2-n_1}{n_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \tag{186}$$

This gives the image distance *i* in terms of the object distance *o*, the outside index of refraction $n_1$ and the properties of the thin lens $r_1, r_2$ and $n_2$. As with mirrors, the focal length of a thin lens is defined to be the image distance when the object distance is very large. Setting *o* = ¥ and writing *f* for the image distance *i*, we obtain

$$\frac{1}{f}=\frac{n_2-n_1}{n_1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \tag{187}$$

This allows us to rewrite (186) as

$$\frac{1}{i}+\frac{1}{o}=\frac{1}{f} \tag{188}$$

This is known as the **lensmaker's formula**. Notice that it is exactly the same as for a spherical mirror. Thus, we immediately see that the lateral magnification is

$$m=-\frac{i}{o} \tag{189}$$

Finally, we define a **converging lens** to be one which has a positive focal length. This lens has $r_1 > 0$ and $r_2 < 0$. It is also called a **positive lens**. A **diverging lens** is defined to be a lens which has a negative focal length. This lens has $r_1 < 0$ and $r_2 > 0$. It is also called a **negative lens**. For any other lens in which both $r_1$ and $r_2$ are positive or negative, the lens is converging or diverging depending on which radius of curvature has the greatest magnitude.