AY 127 – SOLUTIONS TO FINAL EXAM

1. **Short questions.** [24 pts]

(a) [6 pts] Suppose the correlation function of galaxies is approximated by $\xi(r) = (r/r_0)^{-\gamma}$ where r_0 is the correlation length. Two types of galaxies ("sample 1" and "sample 2") have biases b_1 and b_2. What is the ratio of their correlation lengths, $r_{0,1}/r_{0,2}$ in terms of b_1, b_2, and γ?

The correlation function for linear biasing is given by $\xi_g(r) = b^2\xi_m(r) = b^2(r/r_{0,m})^{-\gamma}$, where $r_{0,m}$ is the correlation length for matter. The correlation length $r_{0,g}$ for galaxies is the radius r at which $\xi_g(r_{0,g}) = 1$; solving for $r_{0,g}$ gives

$$b^2\left(\frac{r_{0,g}}{r_{0,m}}\right)^{-\gamma} = 1 \quad \rightarrow \quad r_{0,g} = r_{0,m}b^{2/\gamma}.$$

Then the ratio of biases is

$$\boxed{\frac{r_{0,1}}{r_{0,2}} = \left(\frac{b_1}{b_2}\right)^{2/\gamma}}.$$

(b) [6 pts] Explain why CMB observations do not show giant arcs behind galaxy clusters. (Order of magnitude arguments will suffice.)

The CMB is the most distant "source object" that we observe in the Universe so you might think that it experiences the greatest amount of strong lensing. In fact it does, but since lensing conserves surface brightness a perfectly smooth source does not get lensed. (Think of looking at a piece of paper with a magnifying glass – typed words may get magnified, but if the paper is just a blank white sheet you still see a white sheet.) The smallest features we can see in the CMB are at the scale of $\ell \sim 3500$ or $\pi/\ell \sim 3$ arcmin, since Silk damping wipes out smaller perturbations.

Now imagine a cluster that could produce an Einstein radius of 3 arcmin. We use the rule that

$$\theta_E \sim \sqrt{\frac{4GMD_{\rm LS}}{c^2 D_{\rm OL}(D_{\rm OL}+D_{\rm LS})}},$$

where $D_{\rm OL}$ and $D_{\rm LS}$ are the observer-lens and lens-source distances (I'll ignore the factors of $1 + z_{\rm lens}$ here). So if these distances are both of order c/H_0, as appropriate for a cluster at redshift of order unity, then we find

$$M \sim \frac{\theta_E^2 Dc^2}{2G} \sim \frac{\theta_E^2 c^3}{2GH_0} \sim \frac{(10^{-3})^2(10^{10.5}{\rm cm/s})^3}{10^{0.3}(10^{-7.2}{\rm cm^3/g/s^2})(10^{-17.5}/{\rm s})} \sim 10^{49.9}\,{\rm g} \sim 10^{16.5}M_\odot.$$

There just aren't any clusters that big.

(c) [12 pts] Suppose the dark matter consists of non-interacting (or weakly interacting) spin-$\frac{1}{2}$ particles. Using arguments based on degeneracy pressure, and the properties of the Milky Way's dark matter halo, what can be said about the particle's mass? (Again, you may work to an order of magnitude.)

We recall that for fermions the number of states per unit volume in phase space is given by $dN/d^3x\,d^3p = 2/h^3$. Then the number of states per unit physical volume at momentum less than some $p_{\rm max}$ is this multiplied by $\frac{4}{3}\pi p_{\rm max}^3$, or $dN/d^3x = (8\pi/3h^3)p_{\rm max}^3$. If the DM particle is a fermion of mass m, then the density $\rho_{\rm DM}$ is bounded by m times this density of states:

$$\rho \leq \frac{8\pi m}{3h^3}p_{\rm max}^3 = \frac{8\pi m^4}{3h^3}v_{\rm max}^3,$$

where $v_{\rm max}$ is the maximum velocity of bound particles. We can thus infer that

$$m \geq \left(\frac{3\rho}{8\pi}\right)^{1/4}\left(\frac{h}{v_{\rm max}}\right)^{3/4}.$$

In the MW halo, the DM particles are traveling near the circular velocity so we might take $v_{\rm max} \sim$ 300km/s$\sim 10^{7.5}$ cm/s, and $h = 10^{-26.2}$ g cm^2/s so $h/v_{\rm max} \sim 10^{-33.7}$ g cm. The density is given by $\rho \sim 1/(Gt_{\rm orb}^2)$; for $t_{\rm orb} \sim 10^8$ yr$\sim 10^{15.5}$ s, we find $\rho \sim 10^{-24}$ g/cm^3. This sets a limit of $m \geq 10^{-32}$ g or – in particle physics units –

$$\boxed{m \geq 10 \text{ eV}/c^2}.$$

2. **Clusters and CMB polarization.** [30 pts]
Consider a galaxy cluster of virial mass $M_{\rm vir} = 5 \times 10^{14} M_\odot$ located at $z = 0.1$. Assume a flat ΛCDM universe with $\Omega_m = 0.3$, $\Omega_b = 0.05$, and $H_0 = 70$ km/s/Mpc. You may work in the approximation $z \ll 1$ to avoid doing messy numerical integrals.

(a) [10 pts] What is the virial radius $r_{\rm vir}$ of the cluster in kpc? What is the apparent size of this virial radius in arcmin?

The mean density within the virial radius is ~ 200 times the critical density, which is

$$\rho_{\rm crit} = \frac{3H_0^2}{8\pi G} = \frac{3(2.3\times10^{-18}\text{s}^{-1})^2}{8(3.14)(6.67\times10^{-8}\text{cm}^3/\text{g}/\text{s}^2)} = 9.2\times10^{-30}\text{g/cm}^3 = 1.4\times10^{11}\,M_\odot/\text{Mpc}^3.$$

So at 200 times this density, i.e. $2.8\times10^{13}M_\odot$/Mpc^3, a $5\times10^{14}M_\odot$ cluster fits in 18.5 Mpc^3 or setting this equal to $\frac{4}{3}\pi r_{\rm vir}^3$, a radius of 1.6 Mpc or $\boxed{1600\,\text{kpc}}$.

At $z \sim 0.1$ the distance is $D \sim cz/H_0 = 430$ Mpc; the angular size is $\theta_{\rm vir} = r_{\rm vir}/D = 0.0038$ radians or $\boxed{13\,\text{arcmin}}$.

(b) [10 pts] Averaged over a disk whose is the virial radius, what is the optical depth due to Thomson scattering within the cluster? (You may assume the baryons are mainly in the form of ionized intracluster gas.)

The optical depth is $\kappa_{\rm T}\Sigma_{\rm b}$, *where* $\kappa_{\rm T} \sim \sigma_{\rm T}/m_p \sim 0.4$ *cm*^{2}*/g is the Thomson opacity (assuming mostly H; for each proton of mass there is one electron of scatterer) and* $\Sigma_{\rm b}$ *is the baryon column density. The total matter column density in the cluster averaged over the disk is*

$$\Sigma_{\rm m} = \frac{M_{\rm vir}}{\pi r_{\rm vir}^2} = \frac{5\times10^{14}\times2\times10^{33}\ {\rm g}}{3.14\times(1600\times3.1\times10^{21}\ {\rm cm})^2} = 0.013{\rm g/cm}^2,$$

and the baryon column density is a factor of $\Omega_{\rm b}/\Omega_{\rm m} \sim 1/6$ *lower, or* $\Sigma_{\rm b} = 0.0022$ *g/cm*^{2}. *This leads to a mean optical depth of* $\boxed{\tau = 9\times10^{-4}}$.

(c) [10 pts] The cluster is expected to exhibit microwave polarization due to scattering of the quadrupole moment of the CMB anisotropy. To an order of magnitude level, what is the polarized signal in μK?

The CMB quadrupole moment has an RMS amplitude of $\sqrt{C_2} \sim 30\ \mu K$. *So we should multiply this by the optical depth (probability of scattering)* $\tau = 0.0009$ *and include a typical fractional polarization of scattering* $f_{\rm pol} \sim 0.5$. *This gives a CMB polarization signal of* $\boxed{0.014\ \mu{\rm K}}$.

3. **Cosmic X-ray background.** [46 pts]

The observed brightness of the cosmic X-ray background (CXRB) is $\nu I_\nu \approx 3\times10^{-10}$ W/m^2/sr.

(a) [4 pts] Compute the corresponding volume energy density. Compare with the energy density of the CMB.

To get the volume energy density per $\ln\nu$, *we should multiply* νI_ν *by* $4\pi/c$, *to account for the total solid angle and to convert flux (energy per unit area per unit* ***time****) to density (energy per unit area per unit* ***length****). This gives* $u_{\rm CXRB} = 1.3\times10^{-17}{\rm J/m}^3 = \boxed{1.3\times10^{-16}{\rm erg/cm}^3}$. *The CMB energy density is instead*

$$u_{\rm CMB} = aT_{\rm CMB}^4 = (7.55\times10^{-15}{\rm erg/cm}^3/{\rm K}^4)(2.7{\rm K})^4 = 4\times10^{-13}{\rm erg/cm}^3,$$

or about 3000 times greater than that in the CXRB.

(b) [13 pts] It is estimated that on average there is a SMBH with $M_{\rm BH} \sim 10^7 M_\odot$ per average ($L \sim L_\star$) galaxy. Assuming that the efficiency of accretion in converting the rest mass into energy was $\sim 10\%$ (i.e. 90% ends up in the SMBH, 10% is radiated away), and that the mean redshift of emission was $\langle z\rangle \sim 2$, compute the energy density today, generated by the making of these SMBHs. Compare it with the numbers for the CXRB and CMB compuated in (a). (Note: you will need to estimate the comoving density of $L_\star$ galaxies today.)

The comoving number density of $L_\star$ galaxies today is ~ 0.01 Mpc^{-3}, and each one is supposed to have radiated 10%/90% of $10^7 M_\odot$ in energy – that's $1.1 \times 10^6 M_\odot c^2$, and it has redshifted by a factor of $1+z \sim 3$. So the energy density today is

$$u_{\rm rad} = \frac{1.1 \times 10^6 M_\odot c^2}{3} (0.01\,{\rm Mpc}^{-3}) = \boxed{2 \times 10^{-16}\,{\rm erg/cm}^3}.$$

So there's at order of magnitude level enough energy to generate the CXRB, but not too much more. There is not enough energy emitted by accretion to generate the CMB, but we already knew that AGNs were not a plausible origin for the latter.

(c) [8 pts] Consider a quasar with $L_{\rm bol} \approx 5 \times 10^{12} L_\odot$ at $z = 2$. For simplicity, assume an Einstein-de Sitter universe with $H_0 = 50$ km/s/Mpc. Compute the luminosity distance to the quasar. (Note: if you dont know how to evaluate or derive this distance quickly, make a reasoned estimate for only 3 points credit, and move on.)

At $z = 2$, the luminosity distance is

$$D_{\rm L} = (1+z)\frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right) = \boxed{1.5 \times 10^4\,{\rm Mpc}}.$$

(d) [5 pts] Compute its absolute and apparent bolometric magnitudes, assuming $M_{\odot\,\rm bol} = 4.8$ mag.

At $5 \times 10^{12} L_\odot$, the quasar is $2.5 \log_{10}(5 \times 10^{12}) = 31.7$ magnitudes brighter than the Sun, so its absolute magnitude is $\boxed{M_{\rm bol} = -26.9}$. The distance modulus is $5 \log_{10}(1.5 \times 10^{10}) - 5 = 45.9$, leading to an apparent magnitude of $\boxed{m_{\rm bol} = 19.0}$.

(e) [6 pts] Assuming that 30% of the entire luminosity of this quasar is emitted in X-rays, compute the observed X-ray flux (in cgs or SI units).

The observed total flux is

$$F = \frac{L}{4\pi D_{\rm L}^2} = \frac{5 \times 10^{12} \times 4 \times 10^{33}\,{\rm erg/s}}{4(3.14)(1.5 \times 10^4 \times 3.1 \times 10^{24}\,{\rm cm})^2} = 7.4 \times 10^{-13} {\rm erg/cm}^2{\rm /s}.$$

The flux in X-rays is 30% of this or $\boxed{2.5 \times 10^{-13}\,{\rm erg/cm}^2{\rm /s}}$. [In SI: 2.5×10^{-16} W/m^2.]

(f) [10 pts] How many such quasars would it take to generate all of the observed CXRB? Compute their projected surface density on the sky (number per deg^2). How does this compare with what you know about the observed surface density of QSOs on the sky?

To generate 3×10^{-10} *W/m^2/sr from sources with a flux of* 2.5×10^{-16} *W/m^2, we need* 1.2×10^6 *sources per steradian or* $\boxed{400\,\mathrm{deg}^{-2}}$. *The number density of optical quasars varies with flux limits, but the photometric selections from e.g. SDSS give several tens of quasars per deg^2 and these are highly incomplete (e.g. they are missing obscured quasars and quasars coincident with the stellar locus).*