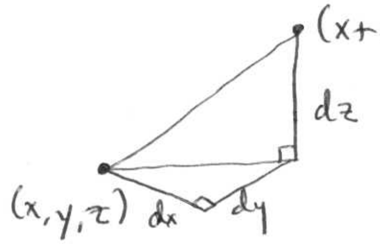
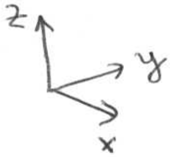
1) Homogeneous, Isotropic Spaces.

First possibility is flat 3D Euclidean space.
Distances given by Pythagorean theorem:



$$dl^2 = dx^2 + dy^2 + dz^2.$$



Can also write in spherical coordinates:

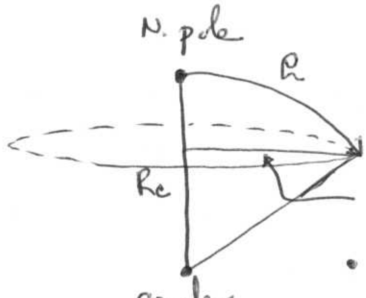
$$dl^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta \, d\phi^2.$$

Both useful:

- perturbation analyses, simulations easiest in Cartesian form.
- observations easiest in spherical form.

Other possibilities for isotropic space? Consider 3-sphere:

$$dl^2 = d\rho^2 + R_c^2 \sin^2 \frac{\rho}{R_c} \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \qquad (\text{"closed space"})$$



(one dimension suppressed).

- Finite. $V = 2\pi^2 R_c^4$.
- No special point on surface
- No special direction.

Alternate forms:

$$dl^2 = \frac{d\chi^2}{1 - R_c^{-2} x^2} + x^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \qquad \text{sub. } x = R_c \sin \frac{\rho}{R_c}.$$

$$dl^2 = \frac{dx^2 + dy^2 + dz^2}{1 + \frac{1}{4R_c^2}(x^2 + y^2 + z^2)} \qquad (\text{stereographic projection}).$$

Also possible to take $R_c^2 \to -R_c^2$ (imaginary radius of curvature!)

$$\Rightarrow \quad d\ell^2 = d\rho^2 + R_c^2 \sinh^2 \frac{\rho}{R_c} \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \qquad (\text{"open space"})$$

- Infinite!
- Still homogeneous & isotropic.

Fourth possibility? Projective sphere. Identify antipodal points on the sphere. Volume now $V = \pi^2 R_c^4$! But no different from sphere unless R_c is small enough to see the "same" object. So we won't talk about this more in this class.

Summary of spaces:

	Euclidean	Closed	Open
Volume	∞	$2\pi^2 R_c^4$	∞
Sum of angles in triangle.	π	$>\pi$	$<\pi$
Pythagorean theorem	$a^2+b^2=c^2$	$a^2+b^2>c^2$	$a^2+b^2<c^2$
Area of circle	$\pi\rho^2$	$<\pi\rho^2$	$>\pi\rho^2$
Volume of sphere	$\frac{4}{3}\pi\rho^3$	$<\frac{4}{3}\pi\rho^3$	$>\frac{4}{3}\pi\rho^3$

2) Robertson-Walker metric.

The Universe can *expand* while remaining homogeneous & isotropic. Suppose distances vary with time according to the scale factor $a(t)$:

$$dl \propto a(t).$$
$$\ell \propto a(t).$$
$$R_c \propto a(t).$$

Let's define:

$\ell = r\, a(t)$ r = comoving radial distance

$R_c = \mathcal{R}\, a(t)$ $\mathcal{R}$ = comoving radius of curvature

Then lengths vary according to:

$$dl^2 = a(t)^2 \left[dr^2 + \begin{Bmatrix} r^2 \\ \mathcal{R}^2 \sin^2 r/\mathcal{R} \\ \mathcal{R}^2 \sinh^2 r/\mathcal{R} \end{Bmatrix} \cdot \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right].$$

(flat, closed, open).

In relativity we more commonly discuss the proper time interval between two neighboring events, which is:

$$ds^2 = \underbrace{dt^2}_{\text{time}} - \frac{dl^2}{c^2} \leftarrow \text{spatial distance}$$

or:
$$ds^2 = dt^2 - \frac{a(t)^2}{c^2} \left[dr^2 + \begin{Bmatrix} r^2 \\ \mathcal{R}^2 \sin^2 \frac{r}{\mathcal{R}} \\ \mathcal{R}^2 \sinh^2 \frac{r}{\mathcal{R}} \end{Bmatrix} \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right].$$

Notes:

- RW metric has a global cosmic time t. Property of FRW universe - not all cosmologies have this.
- Spherical symmetry is manifest. Surface at constant t & constant radial coordinate r is a 2-sphere.
- Definition: $r_1 = \begin{Bmatrix} r \\ \mathcal{R} \sin r/\mathcal{R} \\ \mathcal{R} \sinh r/\mathcal{R} \end{Bmatrix}$ is comoving angular diameter distance.