Ay 127
Midterm Exam

Due: Friday, May 6, 2011 to Chris Hirata's Mailbox (3rd floor Cahill)

TIME LIMIT: **3 hours**

RULES:

- **Closed** books, Internet, etc.
- **Open** course lecture notes, your own notes, and homework sets/solutions **from this course**.
- You may use a calculator (or slide rule, table of logarithms, abacus, etc.).

HINTS:

- The course lecture notes are strongly recommended.
- The three problems will be equally weighted. If you get stuck on one, move on to another. Partial credit will be assigned where appropriate.

#1. Short Questions.

[These are all independent of each other.]

(a) Suppose we attempt to measure the Hubble constant using a standard candle in a galaxy at a distance D. The galaxy has a random peculiar velocity of ~400 km/s. What is the minimum value of D at which the peculiar velocity introduces no more than 10% error in the Hubble constant?

(b) What is the change ΔT in the CMB temperature during the lifetime of a typical astronomer?

(c) One uncertainty in the theoretical prediction of Big Bang Nucleosynthesis yields is the half-life of the neutron t_n (disagreements of ~1% have been reported in laboratory determinations of t_n). Would you expect Y (the abundance by mass of ^{4}He) to be increased or decreased by longer t_n? To order of magnitude level, how much uncertainty does t_n introduce into the prediction of Y? Explain your reasoning.

#2. Reionization.

(a) Assuming $\Omega_b \sim 0.05$, $\Omega_m \sim 0.3$ and $H_0 \sim 70$ km/s/Mpc, and assuming most of the baryons in the Universe are in the form of ionized gas, what is the mean electron density n_{e0} in the Universe today in cm^{-3}?

(b) Suppose that the baryons in the Universe – which became neutral during the recombination epoch – was reionized at some redshift z_r. Prove that the optical depth for a CMB photon to be-scattered after reionization is given by

$$\tau = \frac{n_{e0}\sigma_T c}{H_0} \int_{1/(1+z_r)}^{1} \frac{da}{a^4\sqrt{\Omega_m a^{-3} + 1 - \Omega_m}},$$

assuming that Universe is flat and that its energy density is dominated by matter and Λ.

(c) If $z_r >> 1$ (and Ω_m is of order unity), show that the integral is dominated by the lower limit and can be approximated as $\frac{2}{3}\Omega_m^{-1/2}(1+z_r)^{3/2}$. Numerically evaluate the optical depth for re-scattering if $z_r = 10$.

#3. Density Evolution of Open and Closed Universes.

In class, I claimed that the density of the Universe as a function of time for a Universe that contains only matter (negligible Λ or radiation) and is slightly open or closed ("slightly" meaning $|\Omega_k|<<1$) satisfies

$$\rho_m = \frac{1}{6\pi G t^2}\left[1 - \frac{3}{5}\Omega_k\left(\frac{t}{t_0}\right)^{2/3} + O(\Omega_k^2)\right]. \tag{1}$$

(This is a Taylor expansion in Ω_k.) In class, Equation (1) was used in the analysis of large scale perturbations in the Universe. In this problem you will prove this relation.

(a) Use the Friedmann equation to show that the scale factor satisfies the differential equation

$$\frac{da}{dt} = H_0 a^{-1/2}\sqrt{1 + \Omega_k(a-1)}.$$

(b) Express this result by writing the age of the Universe t in terms of the scale factor a, and Taylor expand in Ω_k to show:

$$t = H_0^{-1}\int \frac{a^{1/2}}{\sqrt{1+\Omega_k(a-1)}}\,da = \frac{2}{3H_0}a^{3/2}\left[1 - \Omega_k\left(\tfrac{3}{10}a - \tfrac{1}{2}\right) + \ldots\right]$$

(c) Derive a formula for the density of the Universe as a function of H_0, Ω_k, and a. Solve it for a to show that:

$$a^{3/2} = H_0\sqrt{\frac{3(1-\Omega_k)}{8\pi G\rho}}.$$

(d) Substitute this into your result from (b), and Taylor expand $\sqrt{1-\Omega_k}$ to show that

$$t = \sqrt{\frac{1}{6\pi G\rho}}\left[1 - \tfrac{3}{10}\Omega_k a + \ldots\right]$$

(e) Explain why – to first order in Ω_k – one may replace a in the above expression with $(t/t_0)^{2/3}$. Then solve for ρ and obtain Eq. (1).