Ay 127 - Homework #1

Problem 1 (40 points)

a) Start from Friedmann equation with $\Omega_\Lambda = 0$: $\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{c^2}{a^2 \mathcal{R}^2}$

Using $\rho_c = \frac{3H^2}{8\pi G}$, $\frac{\rho_0}{\rho_{c,0}} = \Omega_0$, $\rho(a) = \frac{\rho_0}{a^3}$, $\Omega_k = 1 - \Omega = -\frac{c^2}{a^2 R^2 H^2}$

this becomes $\underline{\dot a^2} = a^2\left(\frac{8\pi G}{3a^3}\frac{\rho_0}{\rho_c}\frac{3H_0^2}{8\pi G} + \frac{\Omega_{k,0}}{a^2}H_0^2\right) = \frac{\Omega_0}{a}H_0^2 + \underbrace{\Omega_{k,0}}_{=(1-\Omega_0)}H_0^2$

$$= H_0^2\left(\Omega_0\left(\frac{1}{a} - 1\right) + 1\right)$$

b) Using the substitution $a = \frac{\Omega_0}{2(\Omega_0 - 1)}(1 - \cos(\theta))$

$$\dot a = \frac{\Omega_0}{2(\Omega_0 - 1)}\sin(\theta)\,\dot\theta$$

$$\rightarrow \dot a^2 = \frac{\Omega_0^2}{4(\Omega_0-1)^2}\sin^2(\theta)\,\dot\theta^2 = H_0^2\left(\frac{2(\Omega_0-1)}{1-\cos(\theta)} - \Omega_0 + 1\right)$$

$$\rightarrow \frac{\Omega_0}{2(\Omega_0-1)}\sin(\theta)\underbrace{\dot\theta}_{\frac{d\theta}{dt}} = H_0\sqrt{\Omega_0 - 1}\underbrace{\sqrt{\frac{1+\cos(\theta)}{1-\cos(\theta)}}}_{(\tan(\theta/2))^{-1} = \frac{\sin(\theta)}{1-\cos(\theta)}}$$

$$\Rightarrow (1 - \cos(\theta))\,d\theta = \frac{2H_0}{\Omega_0}\sqrt{\Omega_0 - 1}^{\,3}\,dt$$

$$\rightarrow t = \frac{\Omega_0}{2H_0(\Omega_0-1)^{3/2}}(\theta - \sin\theta) \qquad t = 0 \text{ at } \theta = 0,\ a = 0$$

c) comoving distance between t_1, t_2 = distance a photon can travel during this time:

$$\eta(t_1, t_2) = \int_{t_1}^{t_2} c\frac{dt}{a(t)} = \int_{\theta_1}^{\theta_2} c\,\frac{\Omega_0(1-\cos\theta)}{2H_0(\Omega_0-1)^{3/2}}\,\frac{2(\Omega_0-1)}{\Omega_0(1-\cos\theta)}\,d\theta = c\,\frac{\theta_2 - \theta_1}{H_0\sqrt{\Omega_0 - 1}}$$

Big Crunch occurs at $\theta = 2\pi$

$$\Rightarrow t_{crunch} = \frac{\Omega_0\pi}{H_0(\Omega_0-1)^{3/2}};\quad \eta(0, \text{crunch}) = \frac{2\pi c}{H_0\sqrt{\Omega_0-1}} = 2\pi\mathcal{R}_0$$

$\Rightarrow$ photon can circumnavigate universe exactly once between Big Bang and Big Crunch

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Problem 2 (10 points)

a) $\rho_{c,0} = \frac{3H_0^2}{8\pi G}$, density of photons $\rho_{CMB,0} = \frac{4\sigma}{c^3} T_0^4 = 4.66 \times 10^{-34}\ g/cm^3$

$\rightarrow \Omega_{CMB,0} = \frac{\rho_{CMB,0}}{\rho_{c,0}} = 5.06 \times 10^{-5}$

b) $\rho_{CMB}(a) = \rho_{CMB,0}\, a^{-4}$, $\rho_m(a) = \rho_{m,0}\, a^{-3}$

$\rightarrow$ at $a_{eq} = \frac{\rho_{CMB,0}}{\rho_{m,0}} = 1.687 \times 10^{-4} \hat{=} z_{eq} = 5928$

$\rho_{CMB}(a) = \rho_m(a)$, at $z > z_{eq}$ $\rho_{CMB}(a) > \rho_m(a)$

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Problem 3 (50 points)

age of the universe $t_0 = \frac{1}{H_0}\int_0^1 \frac{1}{(\Omega_m/a + (1-\Omega_m-\Omega_\Lambda) + a^2\Omega_\Lambda)^{1/2}}\, da$

Comoving volume $V = 4\pi c\int_0^\infty \frac{D^2(z)}{H(z)}\, dz = \frac{4\pi c}{H_0}\int_0^\infty \frac{D^2(z)}{(\Omega_m(1+z)^3 + (1-\Omega_m-\Omega_\Lambda)(1+z)^2 + \Omega_\Lambda)^{1/2}}\, dz$

where $D(z) = \begin{cases} r(z) & , \Omega_m + \Omega_\Lambda = 1 \\ \mathcal{R}\sin(r(z)/\mathcal{R}) & , \Omega_m + \Omega_\Lambda > 1 \\ \mathcal{R}\sinh(r(z)/\mathcal{R})) & , \Omega_m + \Omega_\Lambda < 1 \end{cases}$

$$r(z) = \frac{c}{H_0}\int_0^z \frac{1}{(1+z)(\Omega_m(1+z) + 1 - \Omega_m - \Omega_\Lambda + (1+z^2)\Omega_\Lambda)^{1/2}}\, dz$$

$$\mathcal{R} = \frac{c}{H_0}\frac{1}{\sqrt{|\Omega_m + \Omega_\Lambda - 1|}}$$

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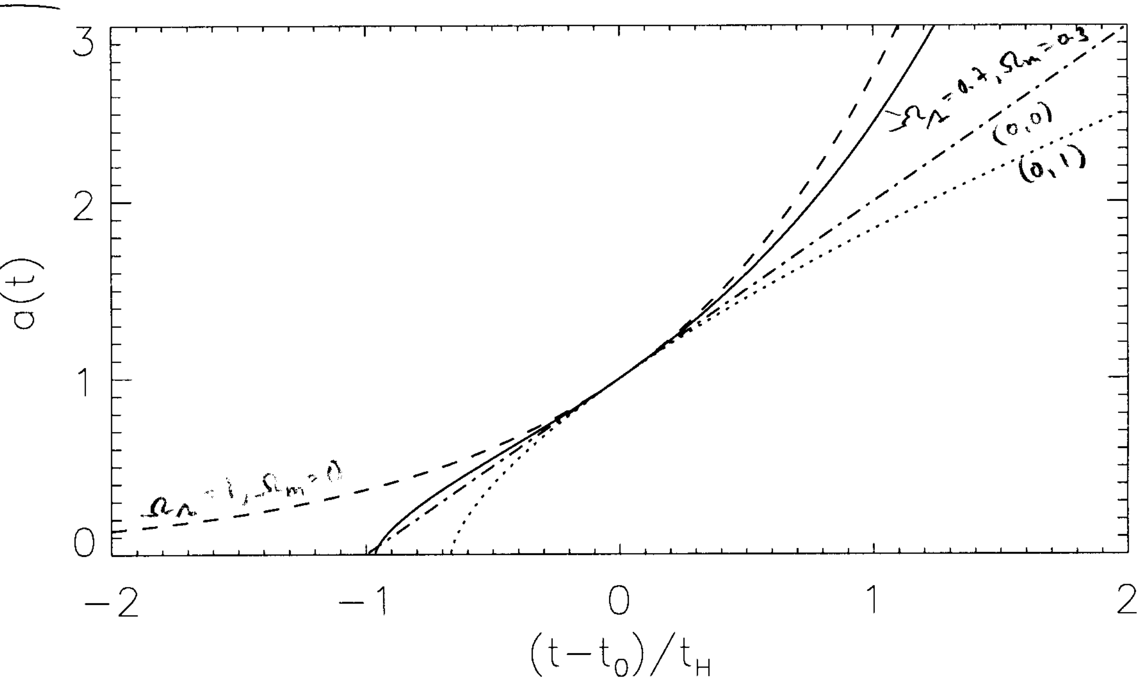
Problem 3

3.d



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Problem 3 (50 points)