

## Ay 127 – Homework 4

[version 2: corrected typo in expression for  $v$  in problem #1a]

Due: Friday, May 31, 2013

Note: This is a technically complicated homework (much harder than what you'll find on the final exam). The point is that after working it you should have a much better understanding of how to solve the types of time-dependent radiative transfer problems that occur in cosmology, high-energy astrophysics, etc.

### 1. Diffusion damping in the CMB – analytical background. [25 points]

In this problem and the following problem, you will estimate the scale at which the CMB power spectrum becomes exponentially damped by diffusion of photons (Silk damping). Such diffusion results in a slow decay of sound waves in the photon-baryon plasma. This problem will also give you some practice with the radiation-hydrodynamical methods that are actually used in CMB codes (albeit with some simplifications).

Let's consider the following idealized problem. Suppose we have a uniform background of photons and baryons, and turn off gravity, dark matter, neutrinos ...; and furthermore, let's assume the energy density is dominated by photons, so that the inertia of the baryons can be neglected. The photons have an integrated intensity  $I(\mathbf{x}, \hat{\mathbf{n}}) = \int_0^\infty I_\nu(\mathbf{x}, \hat{\mathbf{n}}) d\nu$  with units of  $\text{erg}/\text{cm}^2/\text{s}/\text{sr}$ , that depends on position  $\mathbf{x}$  and direction of propagation  $\hat{\mathbf{n}}$ . Our job is to determine how sound waves propagate and damp in this plasma.

We'll consider linear perturbation theory in which there is a single Fourier mode in the  $z$ -direction. Only longitudinal waves will be considered (since these are sourced by density perturbations). Then

$$I(\mathbf{x}, \hat{\mathbf{n}}) = I_0[1 + 4e^{ikz}\Theta(\mu)], \quad (1)$$

where  $\mu = n_z$  is the cosine of the angle between the photon's direction of propagation and the  $z$ -axis. Here  $\Theta(\mu)$  is a small perturbation.<sup>1</sup> Only in problem #2 will we worry about putting this in the expanding Universe; until then we will do everything in physical (as opposed to comoving) units. And for this problem, you may completely neglect polarization and the fact that Thomson scattering is not isotropic (these change the answer by only  $\sim 10\%$ ). This problem should be worked only to linear order in perturbation theory in  $\Theta(\mu)$  and the baryon velocity  $v$ .

(a) We will consider  $I(\mathbf{x}, \hat{\mathbf{n}})$  to represent the intensity in an inertial frame. Show that if the baryons are moving with velocity  $v$  in the  $z$ -direction, then the intensity seen by the baryons is

$$I^{\text{bary}}(\mathbf{x}, \hat{\mathbf{n}}) = I_0 \left[ 1 + 4e^{ikz}\Theta(\mu) - 4\frac{v}{c}\mu \right]. \quad (2)$$

Show that the baryons experience no net force from radiation pressure if

$$v = \frac{3}{2}ce^{ikz} \int_{-1}^1 \mu\Theta(\mu) d\mu; \quad (3)$$

this will be our expression for their velocity since we are neglecting their inertia (i.e. if we neglect  $m$  in  $F = ma$ ).

(b) Suppose the mean free path of the photons is  $\lambda_p$  (due to Thomson scattering: independent of frequency). Show that the general radiative transfer equation gives

$$\dot{I}(\mathbf{x}, \hat{\mathbf{n}}) + c\mu \frac{\partial}{\partial z} I(\mathbf{x}, \hat{\mathbf{n}}) = \frac{c}{\lambda_p} [J^{\text{bary}}(\mathbf{x}) - I^{\text{bary}}(\mathbf{x}, \hat{\mathbf{n}})], \quad (4)$$

where  $J^{\text{bary}}(\mathbf{x})$  denotes the angle-average of  $I^{\text{bary}}$ . [This is very much like the radiative transfer equation you learned about in stellar atmospheres: the main thing you want to explain is why the right-hand side is

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<sup>1</sup>The factor of 4 is conventional: it is because we usually think in terms of fractional temperature perturbations, and the intensity scales as  $T^4$ .

in the baryon frame rather than the inertial frame.] Simplify this to give the result

$$\dot{\Theta}(\mu) + ikc\mu\Theta(\mu) = \frac{c}{\lambda_p} \left[ \frac{1}{2} \int_{-1}^1 \Theta(\mu') d\mu' - \Theta(\mu) + \frac{3}{2}\mu \int_{-1}^1 \mu'\Theta(\mu') d\mu' \right] \equiv \frac{c}{\lambda_p}\epsilon(\mu). \quad (5)$$

Let's define the object in brackets here to be  $\epsilon(\mu)$ .

Also at this point define the integrals  $\Theta_0 = \frac{1}{2} \int_{-1}^1 \Theta(\mu) d\mu$  and  $\Theta_1 = \frac{3}{2} \int_{-1}^1 \mu\Theta(\mu) d\mu$ . Let's take the limit of small  $\lambda_p$ : then  $\epsilon(\mu)$  has to be small, and we may write (with some algebraic rearrangement)

$$\Theta(\mu) = \Theta_0 + \Theta_1\mu - \epsilon(\mu) \quad (6)$$

and

$$\epsilon(\mu) = \lambda_p [c^{-1}\dot{\Theta}(\mu) + ik\mu\Theta(\mu)]. \quad (7)$$

These are the “key equations” that we will use to derive the behavior of sound waves when  $\lambda_p$  is small.

- (c) From Eq. (6), show that  $\int_{-1}^1 \epsilon(\mu) d\mu = 0$  and  $\int_{-1}^1 \mu\epsilon(\mu) d\mu = 0$ .  
 (d) Show that to first order in  $\lambda_p$ :

$$\int_{-1}^1 \left( \mu^2 - \frac{1}{3} \right) \Theta(\mu) d\mu = -\frac{8}{45} ik\lambda_p \Theta_1. \quad (8)$$

[You will need both Eq. (6) and (7) for this.]

- (e) Using the result from parts (c) and (d) and Eq. (7), show that

$$\dot{\Theta}_0 = -\frac{1}{3} ikc\Theta_1 \quad \text{and} \quad \dot{\Theta}_1 = -ikc\Theta_0 - \frac{4}{15} \lambda_p k^2 c\Theta_1. \quad (9)$$

Show that the “zeroeth moment” of the intensity  $\Theta_0$  thus obeys a second-order linear differential equation. It corresponds to a damped harmonic oscillator. What are its frequency  $\omega$  and damping rate  $\gamma$  (i.e. solutions are proportional to  $e^{\pm i\omega t - \gamma t}$ ) as functions of  $k$  – again working only to first order on  $\lambda_p$ ? Give a qualitative/intuitive explanation for how  $\omega$  and  $\gamma$  scale with  $k$ : in particular why these are what you might expect for a diffusively damped sound wave.

## 2. Diffusion damping in the CMB – cosmological context. [20 points]

Now it's time to actually apply the machinery of Problem #1 to the CMB.

(a) Express the physical mean free path  $\lambda_p$  of a photon in terms of redshift  $z$ , ionization fraction  $x_e$ , and cosmological parameters.

(b) Transcribe the results from Problem #1e into a damping rate  $\gamma$  in terms of comoving (instead of physical)  $k$ , but with the correct factors of scale factor  $a$  included. [You shouldn't need to re-derive everything, just use scaling arguments.]

- (c) Show that the number of  $e$ -folds of damping of the amplitude is given by

$$N = \sigma_D^2 k^2, \quad (10)$$

where  $\sigma_D^2 = C \int a^{-2} \lambda_p dt$ . What is  $C$ ? Explain why  $\sigma_D$  is often called the “damping length” or “damping scale.”

(b) From your solution to HW#2 Problem 1b (or from the curve given in the lecture notes), give a rough estimate (a factor of a few is fine) of  $\sigma_D$ . What is the value of  $k$  and hence of multipole  $\ell$  at which the CMB acoustic wave amplitude should be suppressed by one  $e$ -fold?