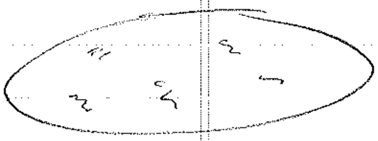
Perturbations

CMB



$\longrightarrow$

Galaxies today

$\frac{\delta\rho}{\rho} \sim 10^{-5}$

$\frac{\delta\rho}{\rho} \sim 10^{6} - 10^{30} - 10^{48}$
(↑ galaxy, ↑ us, ↑ neutron star)

- initially small (linear, $\frac{\delta\rho}{\rho} \ll 1$)
- approx scale free (define later: ~ same ampl in potential on all scales)
- all scales, all times we care about

no expansion - Navier-Stokes:

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \bar{v}\cdot\bar{\nabla}\rho = -\rho\,\bar{\nabla}\cdot\bar{v} \quad \leftarrow \text{continuity}$$

(inviscid, ideal fluid)

$$\frac{Dv}{Dt} = -\bar{\nabla}\Phi - \frac{\bar{\nabla}P}{\rho} \quad \leftarrow \text{momentum}$$

$$\nabla^2\Phi = 4\pi G\rho \quad \leftarrow \text{Poisson}$$

$$\dot{x}_{phys} = \underbrace{\dot{a}\bar{x}_{com}}_{\text{expansion}} + \underbrace{\cancel{\dot{a}}\, a\dot{\bar{x}}_{com}}_{\text{"peculiar velocity"}}$$

expanding: use comoving: $\bar{x}_{phys} = a(t)\,\bar{x}_{com}$

decompose $\bar{V}$: $\bar{V} = \bar{V}_{pec} + \bar{V}_H = \bar{V}_{pec} + a(t)\,\dot{\bar{x}}_{com}$ ← peculiar velocity

$$\dot{a}\bar{x}_{com} = \frac{\dot{a}}{a}(a\bar{x}_{com}) = \frac{\dot{a}}{a}\bar{x}_{phys} = H\,\bar{x}_{phys}$$

~~at origin (x=0)~~

no forces: $\bar{V}_{tot} = \text{const} \rightarrow \dot{\bar{v}}_{pec} = -\dot{a}(t)\,\dot{\bar{x}}_{com} = -H\bar{v}_{pec}$ "Hubble friction" like Force

(at origin, drop $(\dot{a}(t)\,\bar{x})$)

$\bar{F}_H = -H\bar{p}$

(momentum $\propto 1/a$)

subtract ρ_0 from Hubble eqn: $\delta\rho = \rho - \rho_0$

(we're subtracting the 0^{th} order homogeneous solution:
$\rho(\bar{x},t) = \rho_0(t)$, $\bar{v} = H\bar{x}$)

$$\frac{\partial\bar{v}_{pec}}{\partial t} + a^{-1}(\bar{v}_{pec}\cdot\bar{\nabla}_r)\bar{v}_{pec} = a^{-1}\bar{\nabla}_r\Phi - a^{-1}\frac{\bar{\nabla}_r P}{\rho} - H\bar{v}_{pec} \quad \leftarrow \text{must show this}$$

$$a^{-2}\,\nabla_r^2\,\Phi = 4\pi G\,\delta\rho$$

now, define $\delta_m \equiv \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m} = \frac{\delta\rho}{\rho_0}$: know

$$\frac{\partial\bar{\rho}_m}{\partial t} = -3H\bar{\rho}_m \quad (\bar{\nabla}\cdot\bar{v},\ v = H\bar{r})$$

$$\rightarrow \frac{\partial\delta_m}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\rho_m}{\bar{\rho}_m} - 1\right) = \frac{\dot{\rho}_m + 3H\bar{\rho}_m}{\bar{\rho}_m}$$

~~finally, lets assume all perturbs~~

$$\cancel{\frac{\partial \bar v_m}{\partial t} + a^{-1}(\bar v_m\cdot\bar\nabla)\bar v_m = -\frac{\bar\nabla\Phi}{a} - \frac{\bar\nabla p}{a\rho_m} - H\bar v_m}$$

finally, assume all perturbs are small: valid: early everywhere

$\delta_m \ll 1$, $\delta\Phi \ll \Phi_0$, $|\bar v_{pec}| \ll |\bar v_H|$ today $\gtrsim 10\,\mathrm{Mpc}$

~~DM no pressure~~

$$\cancel{\frac{d\delta\bar v}{dt} + H\delta\bar v = -\frac{\bar\nabla\delta p}{\rho_0} - \bar\nabla\delta\Phi}$$

~~i.e.~~

$$\cancel{\frac{d\delta\rho}{dt} + \rho\bar\nabla\cdot\delta\bar v + 3H\delta\rho = 0}$$

$$\rightarrow \frac{\partial\delta_m}{\partial t} = -a^{-1}(\bar\nabla\cdot\bar v_m)$$

$$\frac{\partial\bar v_m}{\partial t} = -H\bar v_m - \frac{\bar\nabla\delta\Phi}{a} - a^{-1}\frac{\bar\nabla\delta p}{\rho_0}$$

DM $\rightarrow 0$ (no pressure)

normal fluid

$$\frac{\partial P}{\partial\rho} = c_s^2\frac{P}{\rho}$$

$$\rightarrow \frac{\bar\nabla\delta P}{\rho_0} \simeq c_s^2\,\bar\nabla\delta_m$$

$$a^{-2}\bar\nabla^2\Phi = 4\pi G\bar\rho_0\,\delta_m$$

~~use plug 1st eqn into~~

take $\bar\nabla\cdot$(mom-eqn) $\rightarrow$ only $\bar\nabla\cdot\bar v_m$ $\leftarrow$ continuity appear

and $\bar\nabla^2\Phi$ $\leftarrow$ Poisson

$\rightarrow$ single ODE: $$\ddot\delta + 2H\dot\delta = \left(4\pi G\rho_0 + \frac{c_s^2\nabla^2}{a^2}\right)\delta$$

Einstein-DeSitter:

$\Omega_m = 1$, DM only

$$\bar\rho_m = \rho_0 = \frac{3H^2}{8\pi G} = \frac{3}{8\pi G}\left(\frac{2}{3t}\right)^2 = \frac{1}{6\pi G t^2}$$

growing

decaying

$$\rightarrow \ddot\delta + \frac{4}{3t}\dot\delta - \frac{2}{3t^2}\delta = 0 \rightarrow \delta = C_1 t^{2/3} + C_2 t^{-1}$$

$$\rightarrow \delta \propto t^{2/3} \propto a$$

continuity $\rightarrow$ $$\bar\nabla\cdot\bar v_m = -a\frac{\partial\delta_m}{\partial t} = \cancel{-aH\delta_m}\; -a\left(\frac{2}{3}\frac{\delta_m}{t}\right) = -aH\delta_m$$

"Growth Function" $G(a)$ such that $\delta_m \propto G(a)$

normalized such that $G(a) = a$ if matter-dominated

$$f(a) = \frac{d\ln G(a)}{d\ln a} = \frac{\dot\delta}{H\delta_m} = \frac{-\bar\nabla\cdot\bar v}{aH\delta_m};$$ EdS $\rightarrow G(a) = a$, $f(a) = 1$

$\Lambda > 0 \rightarrow$ lower $\bar{\rho}_m$, larger H at fixed time

↳ G grows more slowly $\rightarrow f(a) \simeq \Omega_m^{0.6}$

e.g. peculiar vel:

note we can re-write: $\dot{\delta} = \left(\frac{a}{\delta}\frac{d\delta}{da}\right) H \delta = f(a) H \delta$

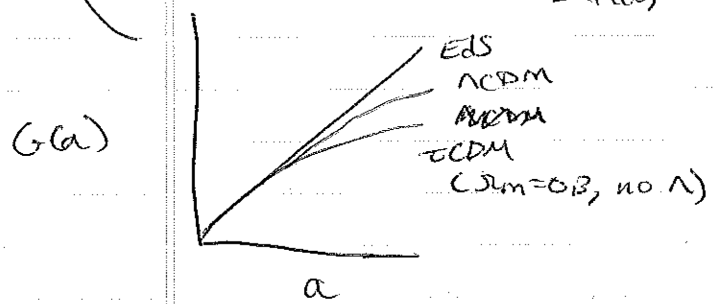
$$\vec{\nabla}\cdot\vec{v} = -aHf(a)\,\delta$$

→ note: curl-free

approximately, ~~[illegible]~~ $f(a) \simeq \Omega_m^{0.6}$

↳ measure overdensities + per. vel of gals on large scales,

$\rightarrow f(a) \rightarrow \Omega_m^{0.6}$



Rad Dom:

~~radiation doesn't "clump"~~ dynamics: $a \propto t^{1/2}$

$H = 1/2t$

$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_r)$

radiation doesn't "clump" $\rightarrow \ddot{\delta} + \frac{1}{t}\dot{\delta} - 4\pi G \bar{\rho}_m \delta_m = 0$

↳ effective $c_s \sim \frac{c}{\sqrt{3}}$ for relativistic fluid

→ Jeans length (in a sec) ≈ horizon

$$\bar{\rho}_m = \frac{\rho_m}{\rho_r}\rho_r = \frac{a}{a_{\rho_m=\rho_r}}\rho_c = \frac{a}{a_{eq}}\rho_c \propto t^{-3/2} \rightarrow \ddot{\delta} + t^{-1}\dot{\delta} - (\,)\,t^{-3/2}\delta = 0$$

early enough times, ρ_m ~~avg~~ term negligible

$\rightarrow \ddot{\delta} + t^{-1}\dot{\delta} = 0$

$$\delta = C_1 + C_2 \ln(t)$$

Baryons: Fourier Space: $\delta(\vec{k}) = \int \delta_m(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d^3r$

Fourier mode: $\delta(\vec{x},t) = \delta_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{r}}$

$$\ddot{\delta}_{\vec{k}} + 2H\dot{\delta}_{\vec{k}} = \left(4\pi G\bar{\rho}_m - \frac{c_s^2 k^2}{a^2}\right)\delta_{\vec{k}}$$

k ~~so~~ large (small scales) → perturb stabilized by pressure

(harmonic oscillator)

no expansion $\rightarrow$

$\delta_k = e^{\pm t/\tau}$ w/ $\tau = (4\pi G\bar{\rho} - c_s^2 k^2)^{-1/2}$

$\rightarrow$ Jeans $\lambda_J = \frac{2\pi}{k_J} = c_s\sqrt{\frac{\pi}{G\rho_0}}$

= critical wavelength

w/ expansion some new behaviors:

~~$k \ll \frac{a}{c_s t}$ $\rightarrow$ like DM~~

~~$k \gtrsim \frac{a}{c_s t}$ $\rightarrow$ pressure dominates RHS~~

Matter era: $\rho_m = \frac{1}{6\pi G t^2}$ $\rightarrow$ $4\pi G\rho_m = \frac{2}{3t^2}$

$$\left(4\pi G\rho_m - \frac{c_s^2 k^2}{a^2}\right) \rightarrow \left(\frac{2}{3t^2} - \frac{c_s^2 k^2}{a^2}\right)$$
$$\rightarrow \left(\frac{2}{3t^2}\right)\left(1 - \frac{k^2}{k_0^2}\right)$$

$$k_0 = \sqrt{\frac{2}{3}}\, a/c_s t$$

~~$k \ll k_0$~~

$k < k_0$ $\rightarrow$ pressure unimpt, behave like DM

$\hookrightarrow$ scale $\lambda \gtrsim$ sound crossing length $\sim \frac{c_s t}{a}$

(*) Power spec — $k > k_0$ $\rightarrow$ pressure FX impt., suppress modes

very long wavelength:

$k < k_H \equiv \frac{aH}{c} \simeq \frac{a}{ct}$ are "superhorizon"

$\hookrightarrow$ light (gravity) can't cross

radiation era $\rightarrow$ $a \propto t^{1/2}$, $H \propto 1/t$

$k_H \propto t^{-1/2}$ $\rightarrow$ as $t \rightarrow 0$, $k_H \rightarrow \infty$ $\rightarrow$ all perturb "started" super-horizon

- causally disconnected on super-horizon, behave as separate Universes

"Super-horizon metric"

$$ds^2 = dt^2 - \frac{a^2(t)}{c^2}\left\{(1 + 2\zeta(\vec{x}))\sum_{i=1}^{3}(dx^i)^2 + 2\sum_{ij=1}^{3} h_{ij}(\vec{x})\, dx^i dx^j\right\}$$

$\sum_{j=1}^{3} \frac{\partial h_{ij}}{\partial x^j} = 0$

ζ = "curvature perturbation", "scalar"

$\hookrightarrow$ these $\rightarrow$ density perturbations $\rightarrow$ scalar, curl-free vel. perturb

h = "tensor perturbations" $\rightarrow$ inflation = from primordial grav. waves $\rightarrow$ no way to make scalar by construction, $\rightarrow$ no density perturb