Physical Cosmology

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Chapter 1

Introduction

Cosmology is the study of the Universe on the largest scales. Up to the 1950s, cosmological data was scarce and generally so inaccurate that Herman Bondi claimed that if a theory did not agree with data, it was about equally likely the data were wrong. Hubble's first determination of the expansion rate of the Universe (Hubble's constant h) was off by a factor of 10. Even as recently at 1995, h was uncertain by 20-40%, depending on who you believed more, and as a consequence most cosmological variables are (still) written with their dependence on h made clear, for example by quoting distances in units of Mpc/h.

Our current cosmological model is based on the solutions to the equations of general relativity, making some general assumptions of isotropy and homogeneity of the Universe at large (which we'll discuss below). These models were all developed in the 1920s, at which stage it had not been fully appreciated that the MW was just one of many million galaxies that make-up the visible Universe. With other words: the assumptions on which the models are based, were certainly *not* inspired nor suggested nor even confirmed by the data at that time. In fact, Einstein's static model was shown to be unstable and so the expansion of the Universe could have been a *prediction* of the theory; surely it would have ranked as one of the most amazing predictions of the physical world based on pure thought. As it happened, Hubble's *observational discovery* of the expansion around the same time relegated the models to describing the data.

Novel observational techniques have revolutionised cosmology over the past decade. The combined power of huge *galaxy redshift* surveys, and *cosmic micro-wave background* (CMB) experiments have lead us into the era of **precision cosmology**, where we start to test the models, and where we can determine their parameters to the percent level. The past years have seen the emergence of a 'standard model' in cosmology, described by around ten parameters. Given how recent this has all happened, we certainly need to keep our minds open for surprises, but the degree to which the models agree with the data is simply astonishing: the current cosmological model is based on general relativity, in which the Universe began in a Hot Big Bang, is presently dominated by dark energy and dark matter, and where the observed structures grew from scale invariant Gaussian fluctuations amplified by gravity. It is called a spatially flat, scale-invariant ΛCDM model, where Λ denotes the cosmological constant (a special case of dark energy), and CDM stands for cold dark matter. A large part of these notes is taken-up by explaining what this all means.

Is this the end of the road? Cosmology is almost unique in the physical sciences that it also demands an answer to the question why the cosmological parameters have the values they do. Is the Big Bang truly a singularity? What happened before that? Is our Universe alone? And do these questions make sense? Not so long ago, most cosmologists would have mumbled that time was created in the BB, and that it therefore made no sense to talk about things which are in principle unobservable, such as other Universes, or anything before the BB. Yet there is currently a flurry of theoretical activity addressing precisely these issues, but it is not clear how we will distinguish different models. The list of questions goes on though, for example *why* are there (at least at the moment) 3+1 dimensions? What about the topology of the Universe? Does it have a simple topology, or could it have the topology of a donut? And do we have any physical theory that even attempts to answer these questions? What is the nature of dark matter? And even more enigmatic, the nature of the dark energy. Particle physics experiments are being designed to look for dark matter particles. What if they never succeed?

These notes describe in more detail the current cosmological model, and the observational evidence for it. Arranged in order of complexity (or detail), try *An Introduction to Modern Cosmology* (Liddle, Wiley 2003), *Physical Cosmology* (Peebles, Princeton 1993), *Cosmological Physics* (Peacock, Cambridge 1999). For more details on the Early Universe, try *Cosmological Inflation and Large-Scale Structure* (Liddle & Lyth, Cambridge 2000) and *The Early Universe* (Kolb & Turner, West View Press, 1990). See also Ned

Wrights tutorial at http://www.astro.ucla.edu/~wright/cosmo_01.htm

Chapter 2

The Homogeneous Universe

2.1 The Cosmological Principle

The Cosmological Principle, introduced by Einstein, demands that we are not in a special place, or at a special time, and that therefore the Universe is **homogeneous** and **isotropic**, *i.e.* it looks the same around each point, and in each direction[1].

This is of course only true on sufficiently 'large scales'. We know that on the scale of the solar system, or the Milky Way galaxy, or on that of the distribution of galaxies around us, that the Universe is neither homogeneous nor isotropic. What we mean, is that, on larger and larger scales, the Universe should become more and more *homogeneous* and *isotropic.*

Does one imply the other? No: for example a Universe with a uniform magnetic field can be homogeneous but is not isotropic. If the distribution of matter around us were a function of distance alone, then the Universe would be isotropic from our vantage point, but it need not be homogeneous. We can (and will) derive the consequences of adopting this 'principle', but the real Universe need not abide by our prejudice!

So we'll start by describing the evolution of, and metric in, a completely homogeneous Universe. Rigour is not our aim here, excellent more rigorous derivations can be found in Peacock's *Cosmological Physics* (Cambridge University Press), or Peebles' *Physical Cosmology* (Princeton University Press).

We are familiar with axioms in mathematics, but the use of a 'principle' in

[1]Note that this was manifestly not true for the known visible Universe in Einstein's time, and even not today for the distribution of optical galaxies.

physics is actually a bit curious. Physical theories are based on observations, and should make testable predictions. How does a 'principle' fit in? The inflationary paradigm (which is not discussed much in these notes) is a physical theory for which the homogeneity of the Universe, as discussed above, is a consequence of a very rapid faster-than-light expansion of the Universe during a very short interval preceding the early classical picture of hot Big Bang (but now m,ore commonly considered to be part of the BB). This is a much healthier description of the Universe on the largest scales, because being a proper physical theory, it is open to experimental testing, in contrast to a principle.

2.2 The Hubble law

Given the Cosmological Principle, what non-static isotropic homogeneous Universes are possible?

Consider two points $\mathbf{r_1}$ and $\mathbf{r_2}$, and let $\mathbf{v}$ be the velocity between them. By homogeneity

$$\mathbf{v}(\mathbf{r_1}) - \mathbf{v}(\mathbf{r_2}) = \mathbf{v}(\mathbf{r_1} - \mathbf{r_2}) \tag{2.1}$$

Hence $\mathbf{v}$ and $\mathbf{r}$ must satisfy a linear relation of the form

$$v_i = \Sigma_j A_{ij} r_j. \tag{2.2}$$

The matrix A_{ij} can be decomposed into symmetric and anti-symmetric parts

$$A_{ij} = A_{ij}^A + A_{ij}^S. \tag{2.3}$$

A_{ij}^A corresponds to a rotation and so can be transformed away by choosing coordinates rotating with the Universe (i.e. non-rotating coordinates).

Then A_{ij}^S can be diagonalised

$$\mathbf{A}^S = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \tag{2.4}$$

and hence $v_1 = \alpha r_1 \quad v_2 = \beta r_2 \quad v_3 = \gamma r_3$. But by isotropy $\alpha = \beta = \gamma = H(t)$.

Hence

$$\mathbf{v} = H(t)\mathbf{r}. \tag{2.5}$$

Note that H has the dimension of inverse time. The current value of Hubble's constant, denoted H_0, is usually written as

$$H_0 = 100\mathrm{h}\,\mathrm{km\ s}^{-1}\mathrm{Mpc}^{-1} \tag{2.6}$$

and for years we did not have an accurate determination of h. From Hubble Space Telescope observations of variable stars in nearby galaxy clusters, we have $h \approx 0.72$, which squares very well with measurements from the CMB. It is not obvious what the uncertainty in the value currently is, because there may be systematic errors in the HST determination, and there are degeneracies in the CMB determination. The Coma cluster, at a distance $r = 90\mathrm{Mpc}$, therefore has a Hubble velocity of around $v = 72 \times 90\mathrm{km\ s}^{-1} = 6480\mathrm{km\ s}^{-1}$.

The recession velocity can be measured via the Doppler shift of spectral lines

$$\Delta\lambda/\lambda = z \approx v/c \quad \text{if } v \ll c. \tag{2.7}$$

This is the definition of the redshift z of an object, but only for nearby galaxies, when $v/c \ll 1$, can you think of it as a Doppler shift.

In the mid-1920s Edwin Hubble measured the distances and redshifts of a set of nearby galaxies and found them to follow this law – **the Hubble Law**. However, his data were far from good enough to actually do this, and the value he found was about 10 times higher than the current one.

The fact that all galaxies seem to be moving away from us does *not* mean we are in a special place in the Universe. You can easily convince yourself that this is now in fact true for all other observers as well. An often made analogy is that of ants on an inflating balloon: every ant sees all the others move away from it, with speed proportional to distance. But as far as the ants, that live on the balloon's *surface* are concerned, no point on the balloon is special.

2.2.1 Fundamental Observers and Mach's principle

Recall from special relativity that the laws of physics are independent of the velocity of the observer. In general relativity this is taken even further: the laws of physics are the same for all observers, also those that are accelerating: these are called freely falling observers. Any physical theory should be formulated to be covariant, that is it transforms in a very specific way between such observers.

We've assumed that the Universe is homogeneous and isotropic – but for which type of observers? Suppose we find a set to which Hubble's law applies. Now consider a rogue observer moving with velocity $\mathbf{v}_p$ with respect to these observers. Clearly, for that observer there *will* be a special direction: along $\mathbf{v}_p$. For galaxies in the observer's direction of motion, v/r will be different from v/r in the opposite direction. Put another way: there is a special velocity in the Universe, namely that in which the mean velocity of all galaxies is zero. Actually, most of the mass may not be in galaxies, or may be the galaxies could be moving with respect the centre-of-mass velocity of the Universe as a whole? This is not as crazy as it may seem.

The Milky Way is falling with a velocity of 175km s^{-1}toward Andromeda. Clearly, for observers in the Milky Way, or in Andromeda, the Hubble law does not strictly apply, as there is this special direction of infall. But the Local Group of galaxies, to which the Milky Way and Andromeda belong, is also falling toward the nearby Virgo cluster of galaxies. So even the Local Group is not a 'good observer' to which the Hubble law applies. But we did not claim that homogeneity and isotropy were valid on any scale: just on 'sufficiently' large scales.

The simply reasoning we followed to 'derive' the Hubble law from homogeneity and isotropy therefore causes us all sorts of problems, and we need to assume that we can attach a 'standard of rest' to the Universe as a whole, based on the visible mass inside it. Note that it is the *velocity* of the reference frame which is 'absolute', not its centre. Observers at rest to this reference frame are called **fundamental observers**. These would be ants that do not move with respect to the balloon. The velocity of observers with respect to such fundamental observers are called peculiar velocities. So this would be the velocity of ants, with respect to the balloon.

There is some circularity in this reasoning: fundamental observers are those to which Hubble's law applies, and Hubble's law is $v = H(t)r$, but only for fundamental observers. It's a bit like inertial frames in Newtonian mechanics. In an inertial frame, $\mathbf{F} = m\mathbf{a}$, and vice versa, inertial frames are those in which this law applies. **Ernst Mach** in 1872 argued that since acceleration of matter can therefore only be measured relative to other matter in the Universe, the existence of inertia must depend on the existence of other matter. There is still controversy whether General Relativity is a Machian theory, that is one in which the rest frame of the large-scale matter distribution is necessarily an inertial frame.

2.2.2 The expansion of the Universe, and peculiar velocities

The typical rms velocity of galaxies with respect to the Hubble velocity, is currently 600km s^{-1}. So only if r is much large than 600km s$^{-1}/H$ can you expect galaxies to follow the Hubble law. We will see that the Local Group moves with such a velocity with respect to the micro-wave background, the sea of photons left over from the BB. Presumably, this peculiar velocity is the result of the action of gravity combined with the fact that on smaller scales, the distribution of matter around the MW is not quite spherically symmetric. Even now, it is not fully clear what 'small scale' actually means.

Note that Hubble's constant, $H(t)$, is constant in space (as required by isotropy), but isotropy or homogeneity do not require that it be constant in time as well. We will derive an equation for its time-dependence soon. Of course, in relativity, space and time are always interlinked (there is a gauge degree of freedom). But our observers can *define* a common time, and hence synchronise watches, by saying $t = t_0$, when the Hubble constant has a certain value, $H = H_0$.

If $v = H(t)r$, then for $r \geq c/H(t)$, galaxies move faster than the speed of light. Surely this cannot be right? In our example of ants on a balloon, it was the inflation of the balloon that causes the ants to think they are moving away from each other, but really it is the stretching of the balloon: no ants need to move at all. Put another way, for small velocities, you can think of the expansion as a Doppler shift. Now in special relativity, when adding velocities you get a correction to the Newtonian $v_{\rm total} = v_1 + v_2$ velocity addition formula, which guarantees you can never get $v_{\rm total} \geq c$ unless a photon, which already had $v = c$ to begin with, is involved. But the velocities we are adding here do not represent objects moving. So there is no special relativistic addition formula, and it is in fact correct to add the velocities. But may be it is better not to think of $H(t)r$ as a velocity at all, since no galaxies need to be moving to have a Hubble flow.

Is it space which is being created in between the galaxies? This would suggest that then also galaxies, and in fact we as well, would also expand with the Universe. We don't, and neither do galaxies. One way to think of the expansion is that galaxies are moving away from each other, because that's what they were doing in the past. Which is not very satisfying, because it rightly asks why they were moving apart in the past. See http://uk.arxiv.org/abs/0809.4573 for a recent discussion by John Peacock.

Another startling conclusion from the expansion is that at time of order $\propto 1/H$, the Universe reached zero size: there was a BB. Actually, whether or not there is a true singularity, and when it happened, depends on how H depends on time. The equations that describe this are called the Friedmann equations.

2.3 The Friedmann equations

One of the requirements of General Relativity is that it reduces to Newtonian mechanics in the appropriate limit of small velocities and weak fields. So we can study the expansion of the Universe on a very small scale, to which Newtonian mechanics should apply, and then use homogeneity to say that actually the result should apply to the Universe as a whole. The proper way to derive the Friedmann equations is in the context of GR, but the resulting equation is identical to the Newtonian one. Almost.

So consider the Newtonian behaviour of a shell of matter of radius $R(t)$, expanding within a homogeneous Universe with density $\rho(t)$. Because of the matter enclosed by the shell, it will decelerate at a rate

$$\ddot{R} = -\frac{GM}{R^2}\,, \tag{2.8}$$

where $M = (4\pi/3)\rho_0 R_0^3$ is the constant mass interior to R. Newton already showed that the behaviour of the shell is independent of the mass distribution outside. In GR, the corresponding theorem is due to Birkhoff.

According to how I wrote the solution, you can see that the density of the Universe was ρ_0 when the radius of the shell was R_0. It's easy to integrate this equation once, to get, for some constant K

$$\left(\frac{\dot{R}}{R_0}\right)^2 = \frac{8\pi G}{3}\rho_0\left(\frac{R}{R_0}\right)^{-1} + K\,. \tag{2.9}$$

Now consider two concentric shells, with initial radii A_0 and B_0. At some later time, these shells will be at radii $A = (R/R_0)A_0$ and $B = (R/R_0)B_0$, since the density remains uniform. Therefore, the distance between these shells will grow as $\dot{B} - \dot{A} = (\dot{R}/R)\,(B - A)$. It is as if the shells recede from each other with a velocity $V = \dot{B} - \dot{A}$, which increases linearly with distance $B - A$, with proportionality constant $H = \dot{R}/R$. This constant (in space!) is Hubble's constant, and we've just derived the equation for its time evolution.

The meaning of the constant K becomes clear when we consider $R \to \infty$: it then becomes the speed of the shell. So, when $K = 0$, the shell's velocity goes eventually to zero: this is called a *critical Universe*, and its density is the critical density, ρ_c,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_c\,, \tag{2.10}$$

If the Universe had the critical density, the matter contained in it is just enough to eventually stop the expansion. (Or put another way: the expansion speed of the Universe is just equal to its escape speed.) By measuring Hubble's constant $H_0 \equiv 100h\mathrm{km\ s^{-1}Mpc^{-1}}$, we can determine the critical density as

$$\begin{aligned}\rho_c &= 1.125\times 10^{-5}h^2 m_{\mathrm{h}}\mathrm{cm}^{-3}\\ &= 1.88\times 10^{-29}h^2\mathrm{gcm}^{-3}\\ &= 2.78\times 10^{11}h^2 M_\odot \mathrm{Mpc}^{-1^3}\end{aligned} \tag{2.11}$$

So the critical density is not very large: it is equivalent to about 10 hydrogen atoms per cubic meter! Note that the density of the paper you are reading this on (or the screen if you're reading this on a computer, for that matter), is around $1\mathrm{g/cm}^3$, or around 29 order of magnitudes higher than the critical density.

However, let us look at it from a cosmological perspective. The mass of the Milky Way galaxy, including its dark halo, is around $10^{12}M_\odot$, say[2]. If the Universe had the critical density, then show that the typical distance between MW-type galaxies should be of order a Mpc, which is in fact not so different from the observed value. So observationally, we expect the Universe to have the critical density to within a factor of a few.

If the constant K is positive, then the final velocity is non-zero, and the Universe keeps expanding (an Open Universe). Finally, for a negative

[2] Recall how we obtained this from the motion of the MW with respect to Andromeda.

constant, the Universe will reach a maximum size, turn around, and start to collapse again, heading for a Big Crunch.

Now, the proper derivation of these two equations needs to be done within the framework of GR. This changes the equations in three ways: first, it teaches us that we should also take the pressure into account (for example the pressure of the photon gas, when considering radiation). Secondly, the constant is kc^2, where $k = 0$ or ± 1 (we might have expected that the speed of light was to surface). And finally, there is another constant, the 'cosmological constant' Λ. Where does this come from?

Peacock's book has a nice explanation for why a cosmological constant surfaces. There is in fact no unique way to derive the equations for GR. The only thing we demand of the theory, is that it be generally co-variant, and reduces to SR in the appropriate limit. So we start by looking at the simplest field equation that satisfies this. The field equation should relate the energy-momentum tensor of the fluid, $T^{\mu\nu}$, to the corresponding metric, $g_{\mu\nu}$. Now there only is one combination which is linear in the second derivatives of the metric, which is a proper tensor, it is called the Riemann tensor, and has four indices, since it is a second derivative of $g_{\mu\nu}$. The Einstein tensor is the appropriate contraction of the Riemann tensor, and the field equation follows from postulating that the Einstein tensor, and the energy-momentum tensor, are proportional. There is no reason not to go to higher orders of derivatives. But in fact, we have ignored the possibility to also look for a tensor which is zero-th order in the derivatives: this is the cosmological constant. Recall that Einstein introduced Λ, because he was looking for a static Universe, and hence needed something repulsive to counter-balance gravity. This does not work, because such a static Universe is unstable. And soon Hubble found that the Universe is expanding – not static. So, there was no longer any need to assume non-zero Λ. Curiously, there is now firm evidence that the Universe in fact has a non-zero cosmological constant. More about that later.

The full equations, including pressure and a cosmological constant, are the famous Friedmann equations,

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho + \frac{kc^2}{a^2} + \frac{\Lambda}{3}\,. \end{aligned} \tag{2.12}$$

I have replaced the radius $R(t)$ of the sphere by the 'scale factor' $a(t)$.

In our derivation, the absolute size of the sphere did not matter, and so we might as well consider just the change in the radius, $R(t)/R(t=t_0) = a(t)$, by choosing $a(t_0) = 1$.

One way to see how these come about, is to introduce the pressure in the equation of motion for the shell, through,

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a\,. \tag{2.13}$$

Now consider the enclosed volume. Its energy is $U = \rho c^2 V$, where $V = (4\pi/3)a^3$ is its volume. If the evolution is isentropic, i.e., if there is no heat transport, then thermodynamics (energy conservation) tells us that

$$\begin{aligned} dU &= -pdV \\ &= \rho dV + V d\rho \end{aligned} \tag{2.14}$$

and, since $dV/V = 3da/a$, $\dot{\rho} = -(p+\rho)\dot{V}/V = -3(p+\rho)\dot{a}/a$. Eliminating the pressure between this equation, and the equation of motion, leads us to $\dot{a}\ddot{a} = (8\pi G/3)(\rho a\dot{a} + \dot{\rho}a^2/2)$. This is a total differential, and introducing the constant K, we get the second of the Friedmann equations (but still without the cosmological constant, of course).

Note that a positive cosmological constant can be seen to act as a repulsive force. Indeed, the acceleration of the shell, can be rewritten as

$$\ddot{a} = -\frac{4\pi G}{3}\rho a + \frac{\Lambda}{3}a \equiv -\frac{GM}{a^2} + \frac{\Lambda}{3}a\,. \tag{2.15}$$

You'll recognise the first term as the deceleration of the shell, due to the matter enclosed within it. The second term thus acts like a repulsive force, whose strength $\propto a$.

2.4 Solutions to the Friedmann equations

We can solve the Friedmann equations (FE, Eqs.2.12) in some simple cases. The equation for the evolution of the Hubble constant $H = \dot{a}/a$ is

$$H^2 = (\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho + \frac{k}{a^2} + \frac{\Lambda}{3} \tag{2.16}$$

So its evolution is determined by the dependence on the matter density (first term), the curvature (second term), and the cosmological constant (last term).

2.4.1 No curvature, $k = 0$

The total density ρ is the sum of the matter and radiation contributions, $\rho = \rho_m + \rho_r$.

For ordinary matter, such as gas or rocks (usually called 'dust' by cosmologists), the density decreases with the expansion as $\rho \propto 1/a^3$, whereas for photons, $\rho \propto 1/a^4$. In both cases, there is an a^3 dependence, just expressing the fact that the number of particles (gas atoms, or photons), in a sphere of radius a is conserved – and hence $\rho \propto 1/a^3$. But for photons, the energy of each photon in addition decreases as $1/a$ due to redshift – and hence $\rho \propto 1/a^4$ for a photon gas (we'll derive a more rigorous version of this soon). So in general, $\rho = \rho_m(a_o/a)^3 + \rho_r(a_o/a)^4$.

For a **matter dominated Universe** ($\rho_m >> \rho_r$, $\Lambda = 0$), we have simplified the FE to

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho_m(\frac{a_o}{a})^3 \tag{2.17}$$

for which you can check that the solution is $a \propto t^{2/3}$. Note that ρ_m is now a constant: it is the matter density when $a = a_o$. Writing the constant as $a = a_o(t/t_o)^{2/3}$, and substituting back into the FE, we get that $t_o = \sqrt{1/6\pi G\rho_m} = 2/3H_0$. This case, a critical-density, matter-dominated Universe, is also called the **Einstein-de Sitter** (EdS) Universe. Note that the Hubble constant $H(z) = H_0(1+z)^{3/2}$. For $H_0 = 100\mathrm{hkm\ s^{-1}Mpc^{-1}}$, $H_0^{-1} = 0.98 \times 10^{10}\mathrm{h^{-1}yr}$

We can do the same to obtain the evolution in the case of **radiation domination**, $\rho = \rho_r(a_o/a)^4$ (hence $\rho_m = \Lambda = 0$). The solution of the FE is easily found to be $a \propto t^{1/2}$ (just substitute an *Ansatz* $a \propto t^\alpha$ and solve the resulting algebraic equation for α), and, again writing the constant as $a = a_o(t/t_o)^{1/2}$, you find that $t_o = \sqrt{3/32\pi G\rho_r} = 1/2H_0$.

Finally, a **cosmological constant dominated Universe** is even easier. Since $(\dot{a}/a)^2 = \Lambda/3$, the Universe expand exponentially, $a \propto \exp(\sqrt{\Lambda/3t})$. Incidentally, this is what happens during inflation.

2.4.2 The Evolution of the Hubble constant

In a critical density, matter dominated Universe, we have $\rho_r = 0$, and $k = \Lambda = 0$, and so $(\dot{a}/a)^2 = 8\pi G\rho_c/3$. The matter density ρ_m is the critical density, ρ_c, and we will denote the Hubble constant now as H_0. So dividing Eq. (2.16) by $H_0^2 \equiv (8\pi G/3)\rho_c$, we get the equivalent form

$$\begin{aligned} H^2 &= H_0^2(\frac{\rho_m/a^3 + \rho_r/a^4}{\rho_c} + \frac{k}{H_0^2 a^2} + \frac{\Lambda}{3H_0^2}) \\ &= H_0^2(\Omega_m/a^3 + \Omega_r/a^4 + \Omega_k/a^2 + \Omega_\Lambda)\,, \end{aligned} \tag{2.18}$$

where we have introduced the constants $\Omega_m = \rho_m/\rho_c$, $\Omega_r = \rho_r/\rho_c$, $\Omega_k = k/H_0^2$ and $\Omega_\Lambda = \Lambda/3H_0^2$. They characterise how much matter, radiation, curvature and cosmological constant, contribute to the total density. Note that, since we have written Hubble's constant at $a = 1$ as H_0, we have by construction, that $\Omega_m + \Omega_r + \Omega_k + \Omega_\Lambda = 1$.

Current best values are $(\Omega_m, \Omega_\Lambda, \Omega_k) = (0.3, 0.7, 0)$, with $\Omega_r \approx 4.2 \times 10^{-5}h^2$ and currently the Universe is dominated by the cosmological constant. At intermediate redshifts ≥ 1 say, we can neglect the cosmological constant, and $H(z) \approx H_0\sqrt{\Omega_m}(1+z)^{3/2}$. Given $H_0 = 70$km s^{-1}Mpc^{-1}, this gives $H(z=3) \approx 307$km s^{-1}Mpc^{-1}. For the Coma cluster, at distance 90Mpc, we found that the current Hubble velocity is $70 \times 90 = 6300$km s^{-1}. At redshift $z = 3$, this would have been $307 \times 90/(1+3) = 6907$km s^{-1}.

The cosmological constant starts being important when $\Omega_m/a^3 \leq \Omega_\Lambda$ which is for $a \geq (\Omega_m/\Omega_\Lambda)^{1/3} \approx 0.75$ or $z \leq 1/a - 1 = 0.3$. This may seem rather uncomfortably close to the present day.

As we study the Universe at earlier and earlier times, $a \to 0$, the cosmological constant, curvature, and matter (in that order, unless the curvature term is zero) become increasingly less important, until eventually only radiation matters in determining $H(a)$: no matter what the values of Ω_m, Ω_k

and Ω_Λ are now, the Universe becomes eventually radiation dominated for sufficiently small a (or high redshift). In this case, $H(t) = H_0\,\rho_r^{1/2}(1+z)^2$. We can compute when, going back in time, radiation starts to dominate the expansion over matter, by computing Ω_r from the CMB (see later): $\Omega_r \approx 4.2\times 10^{-5}h^2$, and hence $\Omega_m >> \Omega_r$ now. From these numbers, we find at redshift $z_{\rm eq}$, the contributions of matter and radiation become equal, where $1+z_{\rm eq} = \Omega_m/\Omega_r \approx 23800\Omega_m h^2$. This is called **equality**.

The various Ωs that occur in Eq. (2.18) are constants: for example $\Omega_m = \rho_m/\rho_c$ now. Because both the matter density and the critical density depend on time, we can also compute how their ratio, $\Omega = \rho/\rho_c$ varies in time. Here, ρ is the total matter density, including radiation and matter. Start from Eq. (2.16), which can be written as

$$H^2 = H^2\Omega + \frac{k}{a^2} + \frac{\Lambda}{3}\,, \tag{2.19}$$

hence

$$\Omega - 1 = -\frac{k}{H^2a^2} - \frac{\Lambda}{3H^2}\,. \tag{2.20}$$

For small a, $H(a)^2 \propto 1/a^4$ when radiation dominates, or at least $H(a)^2 \propto 1/a^3$ when matter dominates. In either case, $1/H^2\,a^2$ becomes small, hence $\Omega(a) \to 1$ irrespective of k or Λ. In the absence of a cosmological constant, $\Omega = 1$ is a special case, since it implies $k = 0$ hence $\Omega(a) = 1$ for all a. This is also true, if we consider $\Omega_\Lambda = \Lambda/3H^2$, in which case

$$\Omega + \Omega_\Lambda - 1 = -\frac{k}{H^2a^2}\,. \tag{2.21}$$

If the Universe were matter (radiation) dominated, then $a(t) \propto t^{2/3}$ ($a(t) \propto t^{1/2}$), and $\Omega + \Omega_\Lambda - 1 \propto t^{2/3}$ ($\propto t$). This is a peculiar result, because it implies that, unless $\Omega+\Omega_\Lambda = 1$, that sum will start to deviate from 1 as time increases: $\Omega + \Omega_\Lambda = 1$ is an unstable solution. For example, since we know that now $\Omega+\Omega_\Lambda \approx 1$, at nucleo-synthesis, when $t \approx 1$ sec, $|\Omega+\Omega_\Lambda-1| \le 10^{-18}$, and the closer we get to the BB, the more fine-tuned $\Omega + \Omega_\Lambda = 1$ needs to be. Because a space with $\Omega + \Omega_\Lambda = 1$ is called flat (see below why), this is also called the flatness problem: unless the Universe was very fine-tuned to have $\Omega_{\rm total} = \Omega + \Omega_\Lambda = 1$ at early times, then we cannot expext $\Omega_{\rm total} \approx 1$ now. Although this is in fact what we find. The easiest way out is to assume that $\Omega + \Omega_\Lambda = 1$ at all times, and hope to find a good theory that explains why. This is for example the case in Inflationary scenario's, which indeed predict that the deviation of $\Omega_{\rm total}$ from 1 is exceedingly small at the end of inflation.

2.5 Evolution of the Scale Factor

At low redshifts, we can develop $a(t)$ in Taylor series, as

$$\begin{aligned} a(t) &= a(t_0) + \dot{a}(t_0)(t-t_0) + \frac{1}{2}\ddot{a}(t_0)(t-t_0)^2 + \cdots \\ &= a_0(1 + H_0(t-t_0) - \frac{1}{2}\,q_0 H_0^2 (t-t_0)^2 + \cdots) \end{aligned} \tag{2.22}$$

where

$$q_0 = -\frac{\ddot{a}_0 a_0}{\dot{a}_0^2} = \frac{\Omega_m}{2} - \Omega_\Lambda \tag{2.23}$$

where we used the fact that radiation is negligible at low z. With $1+z = a_0/a(t)$, we can write this as an expression for the **lookback time** as function of redshift,

$$H_0(t_0 - t) = z - (1 + q_0/2)z^2 + \cdots \tag{2.24}$$

For example for an Einstein-de Sitter Universe, which has $\Omega_{\rm total} = 1 = \Omega_m$, so that $H_0 = 2/3t_0$, and $q_0 = 1/2$

$$\frac{t_0 - t}{t_0} = \frac{3}{2}(z - (1 + 1/4)z^2 + \cdots)\,, \tag{2.25}$$

so that at $z = 1/2$, $t_{1/2} = (23/32)t_0$ or approximately 1/3 of the age of the Universe is below $z = 1/2$.

The more general expression starts from $\dot{a} = aH(a)$, hence $dt = da/aH(a)$ hence

$$H_0 t = H_0 \int_0^a \frac{da}{\dot{a}} = \int_0^a \frac{da}{aE(z)} = \int_z^\infty \frac{dz}{(1+z)E(z)}\,, \tag{2.26}$$

where the function

$$E(z) = \left[\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_k(1+z)^2 + \Omega_\Lambda\right]^{1/2}. \tag{2.27}$$

Here, t is the age of the Universe at redshift z. For $z = 0$, $t = t_0$ and

$$H_0 t_0 = H_0 \int_0^{a_0} \frac{da}{\dot{a}} = \int_0^{\infty} \frac{dz}{(1+z)E(z)}\,. \tag{2.28}$$

Since the Universe is not empty, $\Omega_m > \Omega_r > 0$, this integral converges for $z \to \infty$, which means that the age of the Universe is *finite*, at least if the physics we are using here applies all the way to infinity. This is not necessarily the case: we know that the laws of physics do not apply sufficiently close to the BB. However, this should only occur at exceedingly high energies, long before there were stars, say. So t_0 better be (much?) bigger than the oldest stars we find in the Universe. For a long time, when people assumed $\Omega_m = 1$, mostly because of theoretical prejudice, this was only marginally, and sometimes not even, fulfilled: some Globular Clusters stars were older (as judged from stellar evolution modelling) than the Universe. Since these uncomfortable days, the ages of these stars have come down, whereas the discovery of the cosmological constant has increased t_0, and there is no longer a problem.

Exercise
We will see later why we think the Universe underwent a period of inflation, during which the expansion was dominated by a cosmological constant. Given this, sketch the evolution of the scale factor $a(t)$ from inflation to the present, by assuming that in each subsequent stage of evolution, only one term (the appropriate one!) dominates the expansion. Indicate the redshifts where one period ends and the next one begins. [Hint: plot $a(t)$ vs t in log-log plot] What will happen in the future?

2.6 The metric

A metric determines how distances are measured in a space that is not necessarily flat. In four dimensional space time, the infinitesimal distance is

$$\begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= g_{00} dt^2 + 2 g_{0i} dt dx^i - \sigma_{ij} dx^i dx^j\,. \end{aligned} \tag{2.29}$$

(We'll let Greek indices run from $0 \to 3$, and Roman ones from $1 \to 3$). For two events happening at the same point in space, $dx^i = 0$, and ds is the elapsed time between the two events. Conversely, at a given time, $dt = 0$, ds^2 is minus the distance squared, dl^2, between the two events. Since this distance needs to be positive definite, so does the tensor σ_{ij}. Light travels along geodesics, for which $ds = 0$ (since $dt = dl$ along a light path). (I assume you recall these results from earlier courses.)

Now impose that the Universe described by this metric be homogeneous and isotropic. This requires that $g_{0i} = 0$, since otherwise we've introduced special directions. Now, in our homogeneous Universe, we can get observers to synchronise watches, and determine the time unit, throughout the Universe. (For example by measuring the density, and its rate of change.) Such observers are our 'fundamental observers'. For those, the most general expression for the metric becomes

$$ds^2 = dt^2 - a^2(t)(f^2(r)dr^2 + g^2(r)d\Omega^2)\,. \tag{2.30}$$

In position space, we have chosen spherical coordinates, where r denotes the radial coordinate, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the angular bit, and g and f are to be determined. This is just as in special relativity, except for the scale factor $a(t)$.

To get the spatial part of the metric, we want to derive the general metric in curved space. An elegant way to do so, is as follows. Suppose you want to get the metric on a surface of a (3D) sphere. If we embed that sphere (a two dimensional surface), into 3D space, then we can write down the metric in 3D space, and then impose the constraint that that all points are lying on the surface. Here we'll do the same, except we want to obtain the metric on a 3D surface, and so we'll have to embed it in a four dimensional space. Let's chose Cartesian coordinates in this 4D space, and let's call the radius of the sphere R. Then, using Cartesian coordinates (x, y, z, w), the 3D surface of the 4D sphere is defined by

$$R^2 = x^2 + y^2 + z^2 + w^2 \tag{2.31}$$

and hence $w^2 = R^2 - r^2$, where $r^2 \equiv x^2 + y^2 + z^2$. Taking the differential of the previous equation, we get that

$$RdR = 0 = xdx + ydy + zdz + wdw\,, \tag{2.32}$$

and consequently

$$dw = -\frac{xdx + ydy + zdz}{\sqrt{R^2 - r^2}} = -\frac{rdr}{\sqrt{R^2 - r^2}}\,. \tag{2.33}$$

Introducing again spherical coordinates,

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta\,, \end{aligned} \tag{2.34}$$

we find that the distance on the sphere is

$$dl^2 = dx^2 + dy^2 + dz^2 + dw^2 = (dr^2 + r^2 d\Omega^2) + \frac{r^2 dr^2}{R^2 - r^2}\,. \tag{2.35}$$

The first bit, $dr^2 + r^2 d\Omega^2$, is just the metric in 3D space, the second bit comes about because we are in curved space (in fact, it goes to zero for large curvature, when $R^2 >> r^2$). Rewriting a little bit, we finally get the following result,

$$\begin{aligned} ds^2 &= dt^2 - a^2(t)\left[\frac{dr^2}{1 - k(r/R)^2} + r^2 d\Omega^2\right] \\ &= dt^2 - a^2(t)R^2\left[d\chi^2 + \sin^2(\chi)d\Omega^2\right] \end{aligned} \tag{2.36}$$

where $r/R = \sin(\chi)$ and $k = 1$. This is called a Closed Universe: here is why. Consider the volume of a sphere at time t. The surface of the sphere is $A = 4\pi a^2(t)R^2\sin^2(\chi)$ (the surface is at $d\chi = 0$ and the integral over $d\Omega$ gives the usual 4π) whereas the proper distance between two concentric shells is $dl = a(t)Rd\chi$. Hence the volume is

$$V = \int A dl = 4\pi a(t)^3 R^3 \int_0^\pi \sin^2(\chi)\,d\chi = 2\pi (a(t)R)^3\,. \tag{2.37}$$

So the volume of space in this Universe, at fixed time, is finite, hence why it is called spatially closed.

The metric for an Open Universe can be obtained, by taking $k = -1$. In this case, χ is unbounded and V could be infinite. However, it need not be:

our assumption of homogeneity might break down for large χ, in which case our reasoning would not apply.

Finally, the case of $k = 0$ is called a flat Universe, because the metric in (r, θ, ϕ) corresponds to that of a flat space. However, because the complete metric still has the $a(t)$ multiplicative factor, the *physical* metric is not the same as that for a flat space. We will discuss how to compute distances soon, but first I want to derive the cosmological redshift. The factor $a(t)$ is the scale factor, which describes the rate at which the Universe expands. A little thought will convince you that the equations that describe its evolution, are just the Friedmann equations which we derived earlier.

The line element ds^2 is called the Robertson-Walker line element, after they showed that this is the most general form for the line element in a spatially homogeneous and isotropic space-time, *independent* of general relativity. This is a purely geometric description of the Universe, and does not involve any dynamics. In particular, it does not tell us how the expansion factor evolves in time. We need a dynamical theory for this, for which we used GR so far. So combining the Friedmann equations, that tell us the evolution $a(t)$, with the Robertson-Walker line element, to describe the metric, we have a complete cosmological model. These models are called **Friedmann-Robertson-Walker models**, or FRW models for short. As we shall see later, given the parameters that characterise them, they are extremely successful in describing the cosmological world as inferred from a wide variety of data sets.

Note that it is not obvious at this stage that the integration constant K that appeared in Eq. (2.9) be related to the choice of $k = 0, \pm 1$ in Eq. (2.36), however within GR they are (see e.g. Peebles page 291).

2.7 Cosmological Redshift

Suppose a cosmologically distant source emits two photons, the first one at time t_e, and the second one a short time interval later, at time $t_e + \Delta t_e$. Now choose the axes such that these photons move radially, i.e. $d\theta = d\phi = 0$. Since for the photon $ds = 0$, we have $dt = a(t)dr/\sqrt{1 - kr^2}$ (I assume you've noticed I have been setting the speed of light, $c = 1$ for a while now).

Therefore, introducing r_1, we can integrate along the light ray, to find that

$$\begin{aligned}\int_{t_e}^{t_o} \frac{dt}{a(t)} &= \int_0^{r_1} \frac{dr}{(1-kr^2)^{1/2}} \\ &= \int_{t_e+\Delta t_e}^{t_o+\Delta t_o} \frac{dt}{a(t)}\,. \end{aligned} \tag{2.38}$$

r_1 is the coordinate that corresponds to t_o (the time the photon is observed), and define

$$r_e = \int_0^{r_1} \frac{dr}{(1-kr^2)^{1/2}}\,. \tag{2.39}$$

Note that this measure of distance does not depend on time, because we have taken out the explicit time dependence introduced by the scale factor $a(t)$. It is called the co-moving distance between the points where the photon was emitted and where it is observed. And because it does not depend on time, the second step in Eq.2.38 follows. Rewriting the RHS as

$$\int_{t_e+\Delta t_e}^{t_o+\Delta t_o} = \int_{t_e}^{t_0} + \int_{t_o}^{t_0+\Delta t_o} - \int_{t_e}^{t_e+\Delta t_e}\,, \tag{2.40}$$

we get that

$$\int_{t_o}^{t_o+\Delta t_o} \frac{dt}{a(t)} = \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a(t)}\,. \tag{2.41}$$

and hence for small time intervals

$$\frac{\Delta t_o}{a_o} = \frac{\Delta t_e}{a_e}\,. \tag{2.42}$$

For example, let us take Δt_e the period of the photon (i.e. the time between two successive maxima), and hence the wavelength is Δt_e (since $c = 1$). Equation 2.42 then provides a relation between the emitted and observed wavelength of photon,

$$\lambda_o = \lambda_e(1+z), \qquad ; \quad 1+z \equiv \frac{a_o}{a_e}\,. \tag{2.43}$$

So the ratio of emitted over observed wavelength, equals the ratio of the expansion factors a_e and a_o of the Universe, when the photon was emitted and observed, respectively. This ratio is usually written as $1 + z$, where z is called the redshift of the galaxy that emitted the photon. It is as if the photon's wavelength expands with the Universe. Note that this has *nothing* to do with a Doppler shift; the only thing which is moving is the photon, and it always moves with the speed of light.

Think again of the ants on the balloon. If the ants are moving with respect to the balloon, then in *addition* to the redshift, there will also be a Doppler shift, which adds to the change in wavelength. Because the velocity of the ant is $\ll c$, we can use the non-relativistic Doppler shift equation, and so the observed wavelength $\lambda_o = \lambda_e(1 + z)(1 + v/c)$, where v needs to take account of the velocity of both the ant that observes the light ray, and the one that emits the light ray. For nearby sources, $v/c \ll 1$, but for distant ones most of the change in λ is due to the redshift.

Cosmologists often treat the redshift as a distance label for an object. This is not quite correct: Eq. (2.43) demonstrate that the redshift of a source is not constant in time, since the expansion factor depends on time. We want to know $dz/dt_{\rm obs}$, which, given $1 + z = a_0/a_e$ can be written as:

$$\begin{aligned} \dot{z} &= \frac{dz}{dt_{\rm obs}} = \frac{\dot{a}_0}{a_e} - \frac{a_0}{a_e^2}\frac{da_e}{dt_{\rm em}}\frac{dt_{\rm em}}{dt_{\rm obs}} \\ &= (1+z)H_0 - H(z)\,. \end{aligned} \tag{2.44}$$

Is this measurable? Suppose we observe a source at $z = 3$. Using Eq. (2.18), we find that $H(z = 3) \approx H_0(\Omega_m(1 + z)^3)^{1/2}$, because we can neglect radiation, curvature and the cosmological constant for the current values of the Ωs (see later). Hence $H(z = 3) \approx 4.4H_0$, for $\Omega_m = 0.3$, and $\dot{z} = 0.4H_0 \approx 10^{-18}\mathrm{s}^{-1}$. So over a year, the redshift of the $z = 3$ source changes by $\approx 10^{-11}$. It is not inconceivable that we will one day be able to measure such changes. Note that the 'redshift' would also change if the peculiar velocity of the distant source, or of the observer, would change. And these changes are likely to be far greater than the minuscule $\dot{z}$ which results from the deceleration of the Universe.

2.8 Cosmological Distances

2.8.1 Angular size diameter and luminosity distance

Because of the expansion of the Universe, different measures of distance, which would be the same in a non-expanding Universe, are different. It is not that one is correct and the others are wrong, they are just different. We already encountered the **co-moving distance**

$$r_e = \int_0^{r_1} \frac{dr}{(1-kr^2)^{1/2}} \,. \tag{2.45}$$

Related is the physical or proper distance, $r = a_o r_e$, i.e. the product of the co-moving distance with the expansion factor. Suppose you normalise the expansion factor at a given time, for example now, to be $a_o = 1$. Then proper distance and co-moving distance would be the same. As you go back in time, and hence the expansion factor $a < 1$, then the proper distance decreases, whereas by construction, the co-moving one remains the same. (Hence its name!) In our ants-on-a-balloon analogy, the co-moving distance is the distance between the ants, in units of the radius of the sphere. So it stays the same when the ants do not move wrt the balloon.

For making measurements in the Universe, two other distances are relevant. One way to characterise the distance to an object, is by measuring the angular extent θ of an object with known (physical) size l. You get the angular diameter distance D_A by taking the ratio, $d_A = l/\theta$ (for small θ of course). So by definition of angular size diameter distance, if you increase D_A by a factor two, then the angular extent of the object halves. At the time that the proper size of the object is l, its proper distance is $a_e r_e$, and its angular extent is therefore $\theta = l/(a_e r_e)$. And hence

$$d_A = a_e r_e \tag{2.46}$$

Note that by the time you observe the object – i.e. by the time the light emitted by the object has reached you – its proper distance will have increased by an amount a_o/a_e. And so the angular extent is *not* the ratio of its proper size, l over its proper distance $a_o r_e$, but l/d_A.

Another way to measure distances is to measure the observed flux from an object with known luminosity. The luminosity distance, d_L, is defined such, that if you increase d_L by a factor f, the observed flux will go down like f^2. Suppose the source has intrinsic luminosity $L_e = n_e/\Delta t_e$ – i.e., the source emits n_e photons per time interval Δt_e. Now the observed flux, F_o, – the amount of photon energy passing through unit surface per unit time, is

$$\begin{aligned} F_o &= \frac{n_e}{\Delta t_o}\frac{\lambda_e}{\lambda_o}\frac{1}{4\pi a_o^2 r_e^2} \\ &= L_e \frac{\Delta t_e}{\Delta t_o}\frac{\lambda_e}{\lambda_o}\frac{1}{4\pi a_o^2 r_e^2} \\ &= \frac{L_e}{4\pi a_o^2 r_e^2}\left(\frac{1}{1+z}\right)^2 \\ &\equiv \frac{L_e}{4\pi d_L^2}\,. \end{aligned} \tag{2.47}$$

The number of photons you detect through the surface of a sphere centred on the emitting galaxy, is $n_e/\Delta t_o$, which is not equal to L_e because of time dilation (Eq.2.42). The energy of each photon, $\propto 1/\lambda$, is decreased because of the redshifting of the photons, Eq.2.43. And so the measured energy flux, F_o, is the first line of the previous equation. Using the previous expression for redshift leads to the luminosity distance

$$d_L = a_o r_e (1+z) = d_A (1+z)^2\,. \tag{2.48}$$

Only if the Universe is not expanding, are these equal.

In a non-expanding Universe, the **surface brightness** (SB) of an object does not depend on distance. For an extended source, such as for example a galaxy, the intensity is the luminosity per unit area, $I = L/A$. Surface brightness is the observational version of this, it is the observed flux, dF, per unit solid angle $d\Omega$, $SB = dF/d\Omega$. In Euclidean geometry, the flux decreases $\propto 1/r^2$ but the surface area corresponding to a given solid angle increases $\propto r^2$. Think of the galaxy as made-up of stars of indentical luminosity. For given solid $d\Omega$, the flux received from each star $\propto 1/r^2$ but the number of stars in $d\Omega$ is $\propto r^2$. Equation (2.48) shows this is note the case in the FRW model. Consider a patch of given physical size l on the galaxy. It emits a total flux $I_e \pi l^2$, where I_e is the intrinsic intensity. The corresponding observed flux is

$$
\begin{aligned}
dF_0 &= \frac{I_e \pi l^2}{4\pi d_L^2} \\
&= \frac{I_e d_A^2 d\Omega}{4\pi d_A^2} \frac{1}{(1+z)^4} \\
&\propto \frac{I_e}{(1+z)^4} d\Omega\,. \qquad (2.49)
\end{aligned}
$$

Note that this is independent of the cosmological dependence of $a(t)$, and hence is an important test to eliminate rival models.

For the EdS model, with $k = 0$ and $a(t) = a_0(t/t_0)^{2/3}$, we can easily evaluate the previous integrals. For the co-moving distance, we obtain

$$
r_e = \int_0^{r_1} dr = \int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_{t_e}^{t_o} \frac{dt}{a_o(t/t_o)^{2/3}} = \frac{3t_o}{a_o} \left[1 - (\frac{t_e}{t_o})^{1/3}\right]\,. \qquad (2.50)
$$

Hence the angular-size diameter distance, $d_A = a_e r_e$, is in terms of the redshift, $1 + z = a_o/a_e = (t_o/t_e)^{2/3}$,

$$
d_A = 3t_o \frac{(1+z)^{1/2} - 1}{(1+z)^{3/2}}\,. \qquad (2.51)
$$

This function, plotted in Fig. 2.1, has a maximum for $z = 5/4$. And so, an object of given size, will span a minimum angular size on the sky, when at $z = 5/4$, and will start to appear bigger again when even further away.

How to interpret this? Consider as simple analogy a closed 2D space: the surface of a sphere. Assume you are at at the North pole, and measure the angular extent of a rod, held perpendicular to lines of constant longitude (i.e., parallel to the equator). Now draw lines with constant longitude on the surface, 0 for Greenwich. When the rod is at the equator, it has a certain angular extent, θ, which is a measure of how many lines of constant longitude cross the rod. When the rod moves toward you, it will cross more of the lines: its angular extent increases. But also as the rod moves away from you, its angular extent increases, and becomes infinite at the South pole.

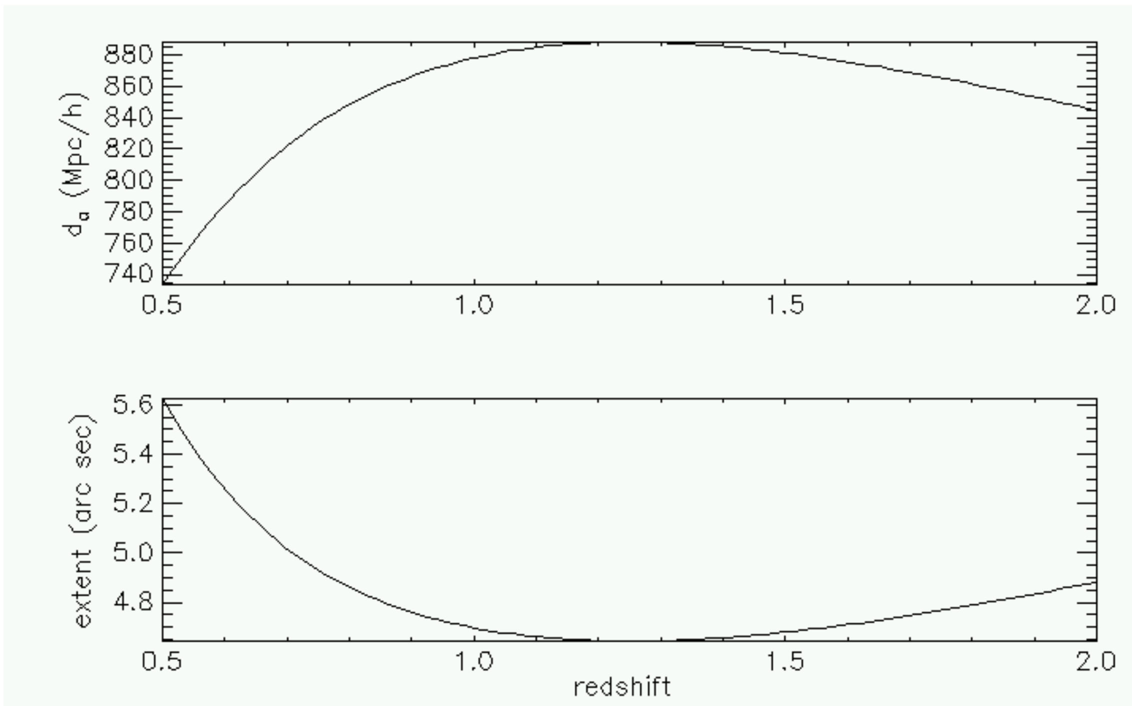

Figure 2.1: *Top panel:* Angular size diameter distance d_A for an Einstein-de Sitter Universe ($\Omega_{\text{total}} = \Omega_m = 1$), as function of redshift. *Bottom panel:* Angular extent θ (in arc seconds) of a galaxy of fixed physical size of 20kpc/h as function of redshift, for the same EdS model. Note how d_A reaches a maximum for $z = 5/4$, and consequently θ a minimum.

The co-moving distance to an object with known redshift, $r_e = \int_0^{r_1} dr/(1-kr^2)^{1/2} = \int_{t_e}^{t_0} dt/a(t)$ depends on how the scale factor varies as function of time, and hence on the parameters of the cosmological model. Suppose we measure the apparent luminosity of objects with the same intrinsic luminosity (a standard candle) as function of z. The apparent luminosity depends on the luminosity distance, and hence also on $a(t)$. This is what the super-nova cosmology project did.

Super novae (SNe) of type I, which are thought to result from mass transfer in binary stars, are also thought to be relatively uniform in their properties, in particular their peak brightness is assumed to be a good standard candle. In addition, these sources are very bright, and so can be observed out to large distances. Figure 2.2 is the apparent magnitude - redshift relation for a sample of SNe, from $z \sim 0$ to $z \approx 1$. Superposed on the data are theoretical curves, which plot the evolution of the apparent brightness for a standard candle, in various cosmological model. In models without a cosmological constant, SN at $z \sim 1$ are predicted to be brighter than the observed ones, hence suggesting a non-zero cosmological constant.

2.8.2 Horizons

Another distance we can compute is the distance that light has travelled since $t = 0$. The co-moving distance r_h (h for horizon) is obtained by taking the limit of $t_e \to 0$ in eq.2.50, $r_h = 3t_o/a_o$. And so the proper size of the horizon is $a_o r_h = 3t_o$. And using our expression for t_o in terms of H_0, the proper size is $2/H_0 = 6 \times 10^3 h^{-1}$Mpc. So, if we lived in an EdS Universe, the furthest distance we could see galaxies to, would be this.

This is a particular case of what is called a 'horizon'. Suppose a galaxy at distance r such that $Hr > c$, emits a light-ray toward us. The proper distance between us and that galaxy is increasing faster than c, and hence the distance between us and the photon is *increasing*: the photon actually appears to move away from us. Depending on how the expansion rate evolves with time, the packet may or may not ever reach us.

Using the metric Eq. (2.36) and without loss of generality, let's put ourselves at $\chi = 0$, and assume the photon moves along the $\theta = \phi = 0$ axis (the Universe is isotropic!). Since for the photon $ds = 0$, $dt = a(t)Rd\chi$, so that

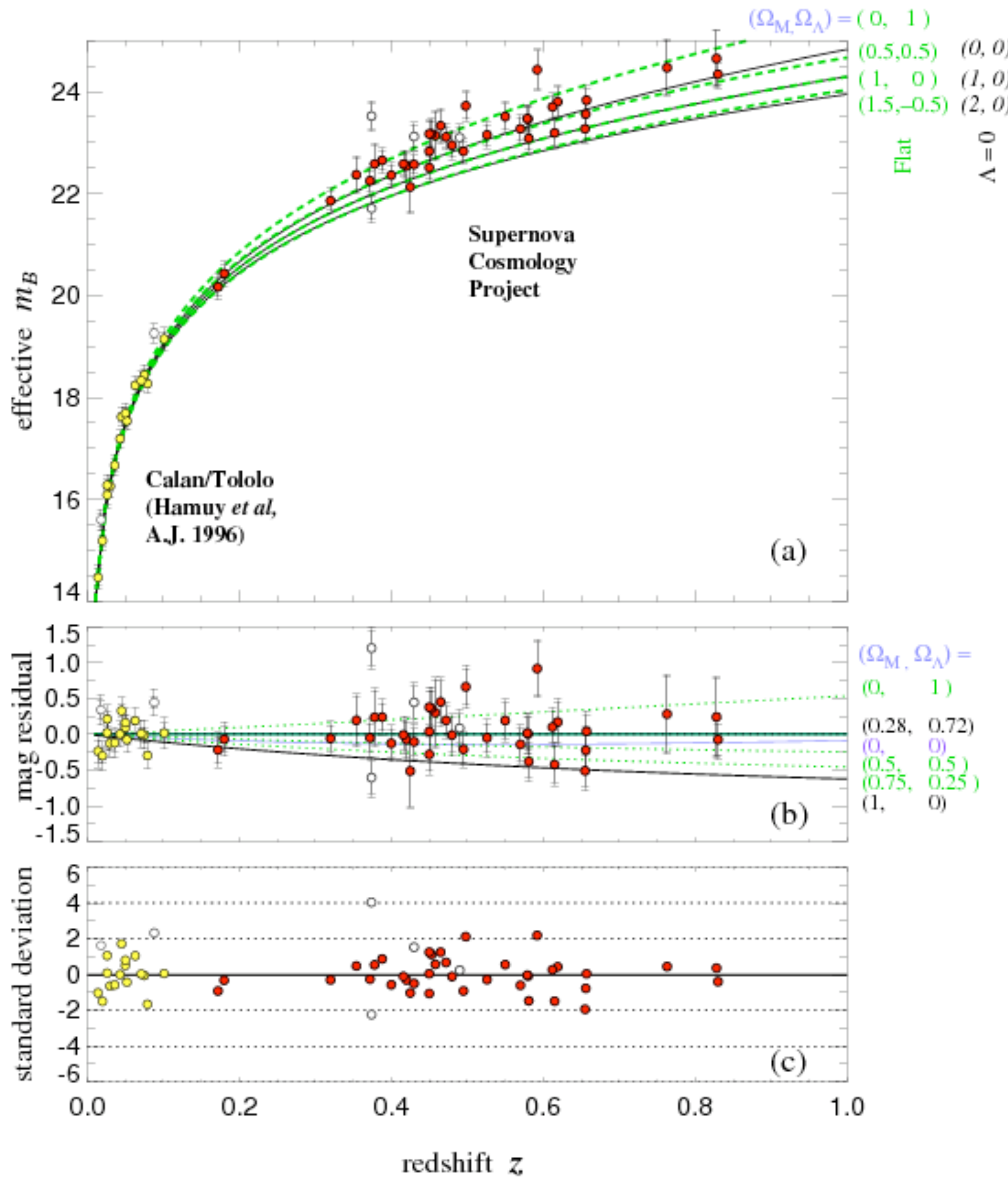

Figure 2.2: Apparent magnitude - redshift relation for super novae (SNe) (symbols with error bars), versus the expected relation based on the luminosity-redshift relation for various models (lines), with the indicated values of the matter density Ω_m and cosmological constant, Ω_Λ. More distant SNe are fainter than expected in an Einstein-de Sitter Universe (which has $(\Omega_m,\Omega_\Lambda)=(1,0)$), but requires a cosmological constant, $(\Omega_m,\Omega_\Lambda)=(0.28,72)$). Figure taken from Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. Journ. 517, 565 (1999)

the co-moving distance travelled by the photon between times t_e and t_0 is

$$R\chi = \int_{t_e}^{t_0} \frac{dt}{a(t)} \,. \tag{2.52}$$

For the EdS case, $k = 0$ and we have $r = R\sin(\chi) \to r = R\chi$, and $a(t) = a_0(t/t_0)^{2/3}$, hence

$$r_e = \int_{t_e}^{t_0} \frac{dt}{a(t)} \,, \tag{2.53}$$

which reduces to $3t_0$ for $t_e \to$, as we had before. Galaxies at greater distances are not causally connected, and are **outside each other's particle horizon**.

Whether or not there is a particle horizon depends on how quickly $a(t) \to 0$ for $t \to 0$: it has to go slower than $a \propto t$ for the integral in Eq. (2.52) to converge for $t_e \to 0$. This is the case for EdS ($a \propto t^{2/3}$) or a radiation dominated Universe ($a \propto t^{1/2}$). So in these cases, two points at opposite directions in the sky, and distance r_e, have never been in causal contact. This begs the question as to why the properties of the CMB, to be discussed below, are so similar. This is called the **horizon problem**.

During inflation when a cosmological constant dominates the expansion, $ds^2 = dt^2 - \exp(2H\Lambda t)(dr^2 + r^2 d\Omega^2)$, and the BB happened for $t \to -\infty$. With $a(t) = \exp(2H_\Lambda t)$, the integral for r_e does **not** converge, and hence there is no particle horizon, and all points will eventually be in causal contact with all other points (at least up to when inflation ends). This might be why the Universe is so strikingly uniform, as causality at least was present at some time in the Universe's past.

2.9 The Thermal History of the Universe

Going back in time, the Universe gets hotter and denser. Therefore, collisions between particles becomes more frequent as well as more energetic, and some of the particles we have now, will be destroyed. A good example of how this works is the cosmic micro-wave background.

2.9.1 The Cosmic Micro-wave Background

Redshifting of a black-body spectrum

The energy density of photons of frequency $\nu_1 = \omega_1/2\pi$ in a black-body spectrum at temperature T_1, is

$$u(\omega_1)d\omega_1 = \frac{\mathrm{h_P}}{2\pi^3 c^3}\frac{\omega_1^3 d\omega_1}{\exp(\mathrm{h_P}\omega_1/2\pi k_\mathrm{B}T_1) - 1}\,. \tag{2.54}$$

Suppose you have such a BB spectrum at some time, $t = t_1$. Then, if no photons get destroyed or produced, at the later time t_0, each photon will appear redshifted according to $\omega_1 = (1+z)\omega_0$, and the distribution will be

$$u(\omega_1)d\omega_1 = (1+z)^4\,\frac{h_\mathrm{P}}{2\pi^3 c^3}\frac{\omega_0^3 d\omega_0}{\exp(h_\mathrm{P}\omega_0/2\pi k_\mathrm{B}T_0) - 1}\,, \tag{2.55}$$

with $T_0 = T_1/(1+z)$. So, still a BB spectrum, but with temperature decreased by a factor $1+z$, and the energy density decreased by a factor $(1+z)^4$ (recall I promised to show that the radiation density scales $\propto 1/a^4 = (1+z)^4$. Note that it was not obvious this would happen. It works, because (1) the exponent is a function of ω/T, and (2) the number of photons $\propto \omega^3 \propto 1/(1+z)^3$.

What this shows is that, even in the absence of interactions, the CMB will retain its BB nature forever.

Origin of the CMB

Figure 2.3 compares the spectrum of the micro-wave background as measured by a variety of balloon and satellite measurements (symbols) to that of a black-body (dotted line). The agreement is amazing.

The measured value of the CMB temperature now is $T = 2.73$K. Since $T \propto (1+z)$, we find that at redshift ≈ 1100, $T \approx 3000$K, and collisions between gas particles were sufficiently energetic to ionise the hydrogen in the Universe. So, at even higher redshifts, the Universe was fully ionised. The beauty of this is, that then the Universe was also opaque, because photons kept on scattering off the free electrons. And so photons and gas were tightly coupled. But we know what the result of that is: the gas will get into a Maxwell-Boltzmann distribution, and the photons into a BB distribution, with the same temperature. So, no matter what the initial distributions

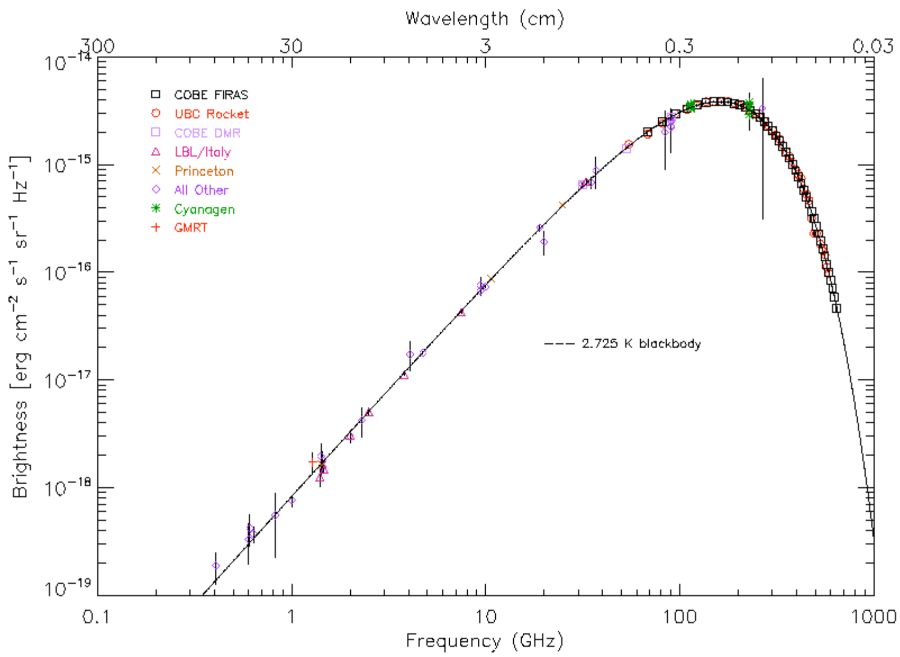

Figure 2.3: Spectrum of the micro-wave background, as measured by a variety of experiments, compared to a Black-Body spectrum. The agreement is astounding.

were, the many interactions will automatically ensure that a BB distribution for photons will be set-up, at least given enough time. And after interactions cease at $z \approx 1100$, because the Universe becomes neutral, the expansion of the Universe ensures that the photon distribution remains a BB distribution, even though gas and photons decouple.

Gamov predicted the existence of relic radiation from the BB, but, while Dicke and collaborators where building instruments to actively look for it, Penzias & Wilson in 1965 discovered a background of micro-wave photons that was exceedingly homogeneous and which they traced to cosmological origins. So they discovered the CMB by accident. In a sense, the CMB sky is reminiscent of Olber's paradox. In Euclidean geometry, surface brightness is conserved, and hence one would expect the night sky to be infinitely bright, clearly in conflict with observations. Applied to the CMB, it is the redshifting of surface brightness $\propto 1/(1+z)^4$ that prevents us being blinded by photons coming from the BB.

The energy density u_r of photons with temperature T is $u_r = a\,T^4$, where $a = 7.5646 \times 10^{-16}$ J m^{-3} K^{-4} is the radiation constant. The corresponding mass density is $\rho_r = u_r/c^2$, hence

$$\Omega_r = \frac{\rho_r}{\rho_c} = \frac{8\pi}{3}\frac{GaT^4}{H_0^2 c^2} \approx 2.5 \times 10^{-5} h^{-2}\,. \tag{2.56}$$

This differs from the usually quoted value of $\Omega_r = 4.22 \times 10^{-5} h^{-2}$, that we have been using so far, because we have neglected the contribution from neutrinos.

Recombination

How does the transition from fully ionised, to fully neutral occur? At high density and temperature, ionisations and recombinations will be in equilibrium. As the Universe expands and cools, the typical collision energy will be unable to ionise hydrogen, and the gas will become increasingly neutral. However, and this is crucial, the rate at which hydrogen recombines *also* drops, since the recombination rate $\propto \rho^2 \propto (1+z)^6$. So a small, but no-zero residual ionisation is left over.

When doing this in more detail, we find two results. Firstly, you would expect ionisation to end, when the typical kinetic energy of a gas particle is equal to the binding energy of the hydrogen atom, $k_B T \approx 13.6$ eV, hence $T \approx 10^4$ K. In fact, because the particles have a Boltzmann distribution, and hence a tail toward higher energies, ionisations only stop when $k_B T \approx 13.6/3$, or $T \approx 3000$ K. Secondly, the gas does not fully recombine, because the recombination rate effectively becomes larger than the age of the Universe. Note that as the gas becomes increasingly neutral, the tiny fraction of free electrons simply cannot find the equally tiny fraction of protons to combine with.

2.9.2 Relic particles and nucleo-synthesis

The process whereby electrons and protons recombine at $z \approx 1100$ leaving a low level of residual ions is typical for how the cooling Universe leaves relic particles. At sufficiently high T, each reaction (e.g. ionisation) is in approximate equilibrium with the opposite reaction (recombination), and the particles are in equilibrium. As T changes, the equilibrium values change, but because the reaction rate is so high, the particles quickly adapt to the new equilibrium. Hence the particles' abundances track closely the slowly changing equilibrium abundance.

However, many of the reaction rates increase strongly with both T and ρ. Consequently, as the Universe expands and cools, the reaction rates drop, and eventually, the rates can not keep-up with the change in equilibrium required. It is often a good approximation to simply neglect the reactions once the reaction rate Γ drops below the age of the Universe $\sim 1/H$, and to assume that the abundance of the particles remains frozen at the last values they had, when $\Gamma \approx 1/H$.

As an example, let us consider the abundance of neutrons and protons. At sufficiently high T, the reactions for converting p to n are

$$\begin{aligned} n + \nu_e \quad &\to p + e^- \\ n + e^+ \quad &\to p + \bar{\nu}_e \, . \end{aligned} \tag{2.57}$$

When in equilibrium, the ratio of ns over ps is set by the usual Boltzmann factor, which arises because n and p have slightly different mass, $N_n/N_p \approx \exp(-(m_n - m_p)c^2/\mathrm{k_B}T)$.

Calculations show that the reaction rate of the reaction in Eq. (2.57) effectively drops below the Hubble time, when $\mathrm{k_B}T \approx 0.8\mathrm{MeV}$ at which time $N_n/N_p \approx \exp(-1.3/0.8) \approx 1/5$, where $(m_n - m_p)c^2 = 1.3\mathrm{MeV}$The relative abundance of N_n/N_p would from then on remain constant, except that free neutrons decay, with a half-life $t_{\mathrm{half}} \approx 612\mathrm{s}$.

When did this happen? At late time, when matter dominates the expansion rate, $a = (t/t_0)^{2/3}$, and hence $T = T_0(t_0/t)^{2/3}$. This is (approximately) valid until matter-radiation equality which is, assuming $t_0 = 12G\mathrm{yr}$,

$$
\begin{aligned}
t_{\mathrm{eq}} &= \frac{t_0}{(23800\Omega_m h^2)^{3/2}} \approx 3300\,(\Omega_m h^2)^{-3/2}\mathrm{yr} \\
T_{\mathrm{eq}} &= 6.5 \times 10^4\,\Omega_m h^2\,\mathrm{K}\,, \qquad (2.58)
\end{aligned}
$$

where I used $z_{\mathrm{eq}} \approx 23800\Omega_m h^2$. Before this time, we can assume radiation domination, hence $a \propto t^{1/2}$, and finally

$$
\begin{aligned}
T &= T_{\mathrm{eq}}\,(\frac{t}{t_{\mathrm{eq}}})^{1/2} \approx 2 \times 10^{10}\mathrm{K}\,\Omega_m h^2 \\
\mathrm{k_B}T &\approx 2\mathrm{MeV}\,\Omega_m h^2 (t/s)^{-1/2}\,. \qquad (2.59)
\end{aligned}
$$

So the reactions coverting n to p 'froze-out' at a time $t \approx 6.3(\Omega_m h^2)^2\mathrm{s}$.

After these reactions ceased, neutrons and protons reacted to produce heavier elements, through

$$
\begin{aligned}
p + n &\rightarrow \mathrm{D} \\
\mathrm{D} + n &\rightarrow {}^3\mathrm{He} \\
\mathrm{D} + \mathrm{D} &\rightarrow {}^4\mathrm{He}\,. \qquad (2.60)
\end{aligned}
$$

Given the corresponding nuclear-reaction rates, one finds that the corresponding destruction reactions (with the opposite arrows) became unimportant below $\mathrm{k_B}T_{\mathrm{He}} \approx 0.1\mathrm{MeV}$. Hence the ^{4}He abundance can be estimated by requiring that all neutrons that were still around at $T = T_{\mathrm{He}}$ ended up inside a Helium atom (and not in Deuterium say). The corresponding time is

$$t_{\rm He} = \frac{2{\rm MeV}(\Omega_m h^2)^2}{0.1{\rm MeV}} \approx 400(\Omega_m h^2)^2 {\rm s}\,. \tag{2.61}$$

At $t = t_{\rm He}$, the neutron over proton abundance had fallen to

$$\frac{N_n}{N_p} = \frac{1}{5}\exp(-400 \times \ln(2)/612) \approx 1/8\,, \tag{2.62}$$

due to spontaneous neutron decay. Because Helium contains 2 neutrons, the Helium abundance by mass is

$$\frac{4(N_n/2)}{N_n + N_p} = \frac{2}{9} \approx 0.22\,. \tag{2.63}$$

A correct calculation of this value also provides the corresponding Deuterium abundance, which can be compared against observations. Deuterium is an espicially powerful probe, because stars do not produce but only destroy Deuterium. The abundance of elements thus produced depends on the baryon density $\Omega_b h^2$ (because the reaction rates do), and one can vary Ω_b to obtain the best fit to the data. The abundances of the various elements depend on Ω_b in different ways, as illustrated in Fig. 2.4, and it is therefore not obvious that BB synthesis would predict the observed values. The fact that it does is strong confirmation of the cosmological model we have been describing.

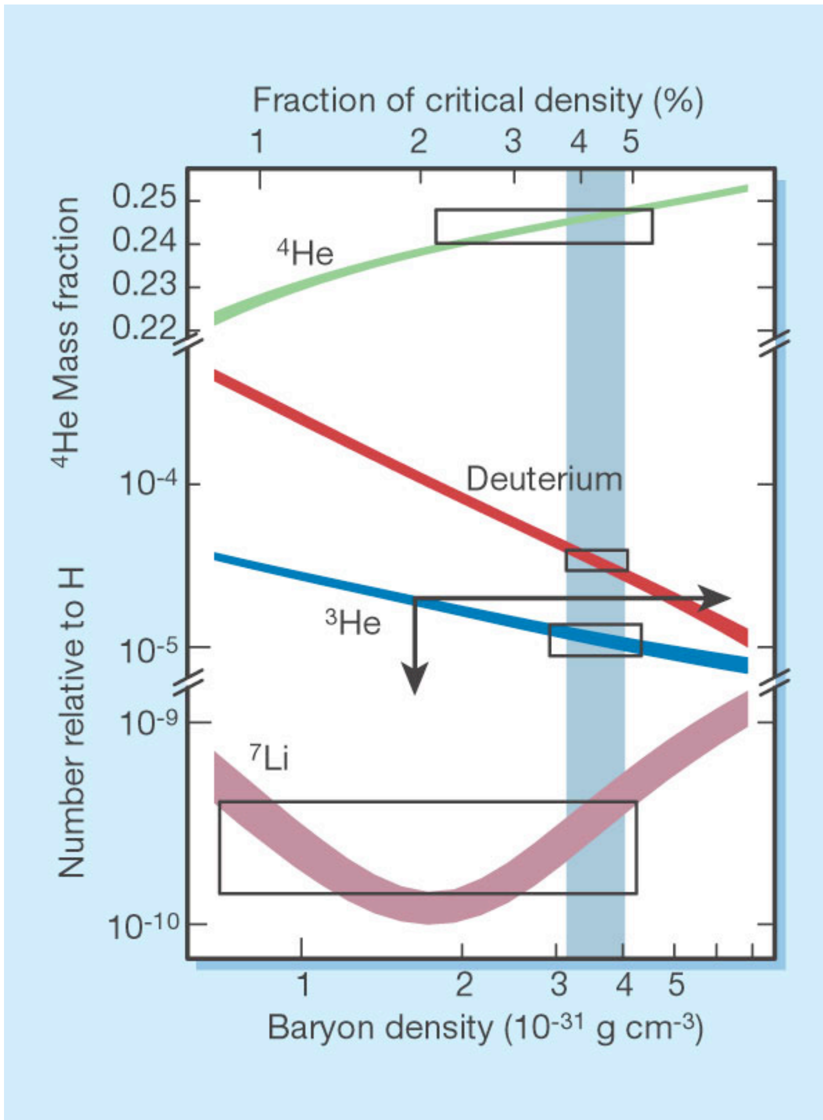

Figure 2.4: Abundance of various elements produced during BB nucleosynthesis as function of the baryon density (curves), versus the observational constraints (vertical band). The very fact that there is a reaonable value of the baryon density that fits all constraints is strong evidence in favour of the Hot Big Bang.

Chapter 3

Linear growth of Cosmological Perturbations

3.1 Introduction

The Cosmology Principle, introduced by Einstein, states that the Universe is **homogeneous** and **isotropic**. As we discussed before, this only applies to 'sufficiently' large scales: on smaller scales, matter is seen to cluster in galaxies, which themselves cluster in groups, clusters and super-clusters. We think that these structures grew gravitationally from very small 'primordial perturbations'İn the inflationary paradigm, these primordial perturbations were at one stage quantum fluctuations, which were inflated to macroscopic scales. When gravity acted on these small perturbations, it made denser regions even more dense, and under dense regions even more under dense, resulting in the structures we see today.

Because the fluctuations were thought to be small once[1], it makes sense to treat the growth of perturbations in an expanding Universe in linear theory. There are several excellent reviews of this, for example see Efstathiou's review in 'Physics of the Early Universe' (Davies, Peacock & Heavens), 'Structure formation in the Universe'(T Padmanabhan, Cambridge 1993).

[1]This need not be the case: structure may have been seeded non-perturbatively as well. The tremendous confirmation of the inflationary paradigm by the CMB data have made these cosmic strings models less popular.

3.2 Equations for the growth of perturbations

We will derive how perturbations in a self gravitating fluid grow when their wavelength is larger than the Jeans length. We will see that the expansion of the Universe modifies the growth rate of the perturbations such that they grow as a power-law in time, in contrast to the usual exponential growth. This is of course a very important difference, and is crucial ingredient for understanding galaxy formation. Mathematically, the difference arises because the differential equations in co-moving coordinates have explicit time dependence that the usual equations do not have. We will assume Newtonian mechanics, as the general relativistic derivation gives the same answer..

The equations we need to solve, those of a self-gravitating fluid, are the continuity, Euler, energy and Poisson equations. They are respectively

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) &= 0 && (3.1)\\
\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} &= -\frac{1}{\rho}\nabla p - \nabla\Phi && (3.2)\\
\rho\frac{\partial}{\partial t}u + \rho\,(\mathbf{v}\cdot\nabla)u &= -p\nabla\mathbf{v} && (3.3)\\
\nabla^2\Phi &= 4\pi G\rho\,. && (3.4)
\end{aligned}
$$

Here, u is the energy per unit mass, $u = p/(\gamma-1)\rho = \mathrm{k_B}T/(\gamma-1)\mu m_\mathrm{h}$.

In the above formulation, the equations describe how the fluid moves with respect to our fixed set of coordinates $\mathbf{r}$. This is called the *Eulerian description*. In the *Lagrangian description*, we imagine ourselves to move along with the fluid, and we want to describe how the fluid's properties such as density, changes along the trajectory. This is accomplished by introducing the Lagrangian derivative, $df/dt \equiv \partial f/\partial t + (\mathbf{v}\cdot\nabla)f$, for any fluid variable f.

In order to take into account the expansion of the Universe, we will write the physical variables $\mathbf{r}$ (the position) and $\mathbf{v}$ (velocity) as perturbations on top of a homogeneous expansion, by introducing the *co-moving position* $\mathbf{x}$ and the *peculiar velocity* $\mathbf{v}_\mathrm{p}$ as

$$\begin{aligned}\mathbf{r} &= a(t)\mathbf{x} \\ \mathbf{v} &= \dot{\mathbf{r}} = \dot{a}\mathbf{x} + a\dot{\mathbf{x}} \equiv \dot{a}\mathbf{x} + \mathbf{v}_{\rm p}\,. \end{aligned} \quad (3.5)$$

Here, $a(t)$ is the scale factor of the expanding (background) cosmological model. The velocity $\mathbf{v}$ is the sum of the Hubble velocity, $\dot{a}\mathbf{x} = H\mathbf{r}$, where $H = \dot{a}/a$ is Hubble's constant, and a 'peculiar' velocity, $\mathbf{v}_{\rm p} = a\dot{\mathbf{x}}$.

Next step is to insert this change of variables into Eqs.3.4. We have to be a bit careful here: the derivative $\partial/\partial t$ in those equations refers to derivatives with respect to t, at constant $\mathbf{r}$. But if we want to write our fluid fields in terms of $(\mathbf{x}, t)$, instead of $(\mathbf{r}, t)$, then we need to express that, now, we want time derivatives at constant $\mathbf{x}$, and *not* at constant $\mathbf{r}$. Suppose we have some function $f(\mathbf{r}, t) \equiv g(\mathbf{r}/a, t)$, such as the density, or velocity, for which we want to compute the partial derivative to time, for constant $\mathbf{x}$, and not constant $\mathbf{r}$. We can do this like

$$\begin{aligned}\frac{\partial}{\partial t} f(\mathbf{r}, t)|_{\mathbf{r}} &= \frac{\partial}{\partial t} g(\mathbf{r}/a, t)|_{\mathbf{r}} \\ &= \frac{\partial}{\partial t} g(\mathbf{x}, t)|_{\mathbf{x}} - (H\mathbf{x} \cdot \frac{\partial}{\partial \mathbf{x}}) g(\mathbf{x}, t)\,, \end{aligned} \quad (3.6)$$

So in fact we just need to replace

$$\frac{\partial}{\partial t}|_{\mathbf{r}} \to \frac{\partial}{\partial t}|_{\mathbf{x}} - (H\mathbf{x} \cdot \nabla)\,, \quad (3.7)$$

where from now on, we will always assume that $\nabla \equiv \partial/\partial\mathbf{x}$, hence $\nabla_{\mathbf{r}} = (1/a)\nabla$ and $\nabla^2_{\mathbf{r}} = (1/a^2)\nabla^2$.

The continuity equation then becomes,

$$(\frac{\partial}{\partial t} - H\mathbf{x} \cdot \nabla)\rho + \frac{1}{a}\nabla\rho(\dot{a}\mathbf{x} + \mathbf{v}_{\rm p}) = 0\,. \quad (3.8)$$

Now, we can write the density as

$$\rho(\mathbf{x}, t) = \rho_b(t)(1 + \delta(\mathbf{x}, t))\,. \quad (3.9)$$

where $\rho_b(t)$ describes the time evolution of the unperturbed Universe, and δ the deviation from the homogeneous solution, which need not be small, but obviously $\delta \geq -1$

In terms of these new variables, the continuity equation becomes

$$(\frac{\partial}{\partial t} - H\mathbf{x}\cdot\nabla)\rho_b(1+\delta) + \frac{1}{a}\nabla\rho_b(1+\delta)(\dot{a}\mathbf{x}+\mathbf{v}_{\rm p}) = 0\,, \tag{3.10}$$

Collecting terms which do not depend on the perturbation, and those that do, we find that

$$\begin{aligned} \dot{\rho}_b + 3H\rho_b &= 0 \\ \dot{\delta} + \frac{1}{a}\nabla\left[(1+\delta)\mathbf{v}_{\rm p}\right] &= 0\,. \end{aligned} \tag{3.11}$$

Now the first equation just describes how the background density evolves. The second one is the continuity equation for the density perturbation.

Using the same substitution in the Poisson equation, we find that

$$\frac{1}{a^2}\nabla^2\Phi = 4\pi G\rho_b(1+\delta) - \Lambda\,, \tag{3.12}$$

where I've also inserted the cosmological constant, Λ. Now let's define another potential, Ψ, as

$$\Phi = \Psi + \frac{2\pi}{3}G\rho_b a^2 x^2 - \frac{1}{6}\Lambda a^2 x^2\,. \tag{3.13}$$

Taking the Laplacian, we get

$$\nabla^2\Phi = \nabla^2\Psi + 4\pi G\rho_b a^2 - \Lambda a^2\,. \tag{3.14}$$

Comparing Eqs. (3.12) and (3.14), note that $\Phi - \Psi$ solves the Poisson equation for the homogeneous case, $\delta = 0$, whereas Ψ is the potential that describes the perturbation,

$$\nabla^2\Psi = 4\pi G\rho_b a^2\delta\,. \tag{3.15}$$

Fine. Now compute the gravitational force in terms of the new potential Ψ,

$$\nabla\Phi = \nabla\Psi + \frac{4\pi}{3}G\rho_b a^2\mathbf{x} - \frac{1}{3}\Lambda a^2\mathbf{x}\,. \tag{3.16}$$

Next insert this into Euler's equation, Eq. (3.2) to obtain

$$
\begin{aligned}
(\frac{\partial}{\partial t} &- H\mathbf{x}\cdot\nabla)(\dot{a}\mathbf{x}+\mathbf{v}_{\rm p}) + \frac{1}{a}\left[(\dot{a}\mathbf{x}+\mathbf{v}_{\rm p})\cdot\nabla\right](\dot{a}\mathbf{x}+\mathbf{v}_{\rm p}) \\
&= -\frac{1}{a}(\nabla\Psi + \frac{4\pi}{3}G\rho_b a^2\mathbf{x} - \frac{1}{3}\Lambda a^2\mathbf{x}) - \frac{1}{a\rho}\nabla p\,. \qquad (3.17)
\end{aligned}
$$

Working-out the left hand side (LHS) of this equation, taking into account that the time derivative assumes $\mathbf{x}$ to be constant, gives

$$
\begin{aligned}
\mathrm{LHS} &= \ddot{a}\mathbf{x} + \dot{\mathbf{v}}_{\rm p} - H\dot{a}\mathbf{x} - H(\mathbf{x}\cdot\nabla)\mathbf{v}_{\rm p} \\
&\quad + \frac{1}{a}\left[\dot{a}^2\mathbf{x} + \dot{a}(\mathbf{x}\cdot\nabla)\mathbf{v}_{\rm p} + \dot{a}\mathbf{v}_{\rm p} + (\mathbf{v}_{\rm p}\cdot\nabla)\mathbf{v}_{\rm p}\right]\,. \qquad (3.18)
\end{aligned}
$$

Now we need to use the fact that our uniform density, $\rho_b(t)$, and the scale factor, $a(t)$, satisfy the Friedmann equation, so that

$$
\ddot{a} = -\frac{4\pi}{3}G\rho_b a + \frac{\Lambda}{3}a\,. \qquad (3.19)
$$

Combining these two leads use to the equation for the time evolution of the peculiar velocity, $\mathbf{v}_{\rm p}$,

$$
\begin{aligned}
\dot{\mathbf{v}}_{\rm p} + H\mathbf{v}_{\rm p} + \frac{1}{a}(\mathbf{v}_{\rm p}\cdot\nabla)\mathbf{v}_{\rm p} &= -\frac{1}{a}\nabla\Psi - \frac{v_s^2}{a}\frac{\nabla\rho}{\rho} \\
\nabla^2\Psi &= 4\pi G\rho_b a^2\delta\,. \qquad (3.20)
\end{aligned}
$$

To get the pressure term, use $\nabla p = (dp/d\rho)\nabla\rho = v_s^2\nabla\rho$, were v_s is the adiabatic sound speed. This assumes that the gas is polytropic, so that the pressure is a function of density, and $dp/d\rho = v_s^2$.

Alternatively, we can substitute $\mathbf{v}_{\rm p} = a\dot{\mathbf{x}}$ to obtain an equation for $\ddot{\mathbf{x}}$:

$$
\ddot{\mathbf{x}} + 2H\dot{\mathbf{x}} + (\dot{\mathbf{x}}\cdot\nabla)\dot{\mathbf{x}} = -\frac{1}{a^2}\nabla\Psi - \frac{v_s^2}{a^2}\frac{\nabla\rho}{\rho}
$$

Collecting all the above, we find:

$$
\begin{aligned}
\dot{\delta} + \nabla\left[(1+\delta)\dot{\mathbf{x}}\right] &= 0 \\
\ddot{\mathbf{x}} + 2H\dot{\mathbf{x}} + (\dot{\mathbf{x}}\cdot\nabla)\dot{\mathbf{x}} &= -\frac{1}{a^2}\nabla\Psi - \frac{v_s^2}{a^2}\frac{\nabla\rho}{\rho} \\
\dot{u} + 3H\frac{p}{\rho} + (\dot{\mathbf{x}}\cdot\nabla)u &= -\frac{p}{\rho}\nabla\dot{\mathbf{x}} \\
\nabla^2\Psi &= 4\pi G\rho_b\delta a^2\,. \qquad (3.21)
\end{aligned}
$$

So far we have only assumed the gas to be polytropic, with $dp/d\rho = v_s^2$, and that the scale factor $a(t)$ satisfies the Friedmann equation.

In the absence of pressure or gravitational forces, Euler's equation reads

$$
\ddot{\mathbf{x}} + 2H\dot{\mathbf{x}} + (\dot{\mathbf{x}}\cdot\nabla)\dot{\mathbf{x}} = 0\,, \qquad (3.22)
$$

which integrates to[2] $\frac{d}{dt}a^2\dot{\mathbf{x}} = 0$ or $a\dot{\mathbf{x}} \propto 1/a$: because of the expansion of the Universe, peculiar velocities $a\dot{\mathbf{x}}$ decay as $1/a$. Similarly, the energy equation integrates in that case to $a^{3(\gamma-1)}\,u$ =constant, hence $T \propto u \propto 1/a^2 \propto (1+z)^2$ for polytropic gas with $\gamma = 5/3$. This also follows from $\rho \propto (1+z)^3$, and $u \propto \rho^{\gamma-1} \propto (1+z)^{3(\gamma-1)}$.

3.3 Linear growth

We can look for the behaviour of *small* perturbations by linearising equations (3.21):

$$
\begin{aligned}
\dot{\delta} + \nabla\dot{\mathbf{x}} &= 0 \\
\ddot{\mathbf{x}} + 2H\dot{\mathbf{x}} &= -\frac{1}{a^2}\nabla\Psi - \frac{v_s^2}{a^2}\nabla\delta \\
\nabla^2\Psi &= 4\pi G\rho_b\delta a^2\,. \qquad (3.23)
\end{aligned}
$$

By taking the time derivative of the first one, and the divergence of the second one, this simplifies to

$$
\ddot{\delta} + 2H\dot{\delta} - \frac{v_s^2}{a^2}\nabla^2\delta = 4\pi G\rho_b\delta\,. \qquad (3.24)
$$

[2] Here, $d/dt = \partial/\partial t + (\dot{\mathbf{x}}\cdot\nabla)$ is the Lagrangian derivative.

For a self-gravitating fluid in a non-expanding Universe, $a = 1$ and $H = 0$, this reduces to

$$\ddot{\delta} - v_s^2 \nabla^2 \delta = 4\pi G \rho_b \delta \tag{3.25}$$

Substituting $\delta(\mathbf{x}, t) = \exp(i(\mathbf{k} \cdot \mathbf{x} - \omega t))$ yields the dispersion relation

$$\omega^2 = v_s^2 k^2 - 4\pi G \rho_b \,. \tag{3.26}$$

For perturbations with wavelength $\lambda/2\pi < (v_s^2/4\pi G \rho_b)^{1/2}$, this solution corresponds to travelling waves, sound waves with wavelength modified by gravity. As the fluid compresses, the pressure gradient is able to make the gas expand again, and overcome gravity. The critical wavelength $\lambda_J/2\pi = (v_s^2/4\pi G \rho_b)^{1/2}$ is called the Jeans length. Waves larger than λ_J are unstable and collapse under gravity, with exponential growth $\propto \exp(-\omega_I \, t)$ where ω_I is the imaginary part of ω.

But in an expanding Universe, with $H \neq 0$, the behaviour is very different. In particular, exponential solutions, $\delta \propto \exp(i(\mathbf{k} \cdot \mathbf{x} + \omega t))$ are no longer solutions to this equation, because H and a explicitly depend on time.

3.3.1 Matter dominated Universe

For example, let's look at the EdS Universe, where $a \propto t^{2/3}$. Writing the behaviour a bit more explicit,

$$\begin{aligned} a &= a_0 (\frac{t}{t_0})^{2/3} \\ H &= \frac{2}{3t} \\ H^2 &= \frac{8\pi G}{3} \rho_b = \frac{4}{9t^2} \,, \end{aligned} \tag{3.27}$$

where I have also used the second Friedmann equation, Eq.**??**, for $k = \Lambda = 0$. Neglecting the pressure term for a second, and substituting the previous equations, I find that the final equation for the perturbation δ becomes

$$\ddot{\delta} + \frac{4}{3t} \dot{\delta} = \frac{2}{3t^2} \delta \,. \tag{3.28}$$

Substituting the *Ansatz* $\delta \propto t^\alpha$, we get a quadratic equation for α and hence two solutions, $\delta \propto t^{2/3}$ and $\delta \propto t^{-1}$ (i.e. $\alpha = 2/3$ and $\alpha = -1$), and hence the general solution

$$\delta = At^{2/3} + Bt^{-1} = Aa + Ba^{-3/2}\,. \tag{3.29}$$

The first term grows in time, and hence is the growing mode, whereas the amplitude of the second one decreases with time – the decreasing mode. Note that they behave as a power law – a direct consequence of the explicit time dependence of the coefficients of Eq. (3.28). This is very important, because it means that unstable perturbations grow only as a power law – much *slower* that the exponential growth of perturbations in the case of a non-expanding fluid. The reason is that the perturbations needs to collapse *against* the expansion of the rest of the Universe. And hence, it is not so easy for galaxies to form out of the expansing Universe.

Note that the growth rate, $\delta \propto t^{2/3} \propto a \propto 1/(1+z)$. So, as long as the perturbation is linear (recall that we linearised the equations, and so our solution is only valid as long as $|\delta| << 1$) it grows proportional to the scale factor.

Recall from our discussion that the amplitude of the perturbations in the CMB is of order 10^{-5}, at the recombination epoch, $z \sim 1000$. So, the amplitude of these fluctuations now, is about a factor 1000 bigger, and so of order 10^{-2}. This is a somewhat simplistic reasoning, but it does suggest that we do not expect any non-linear structures in the Universe today – clearly wrong. This is probably one of the most convincing arguments for the existence of dark matter on a cosmological scale.

3.3.2 $\Omega \ll 1$, matter dominated Universe

For an Open Universe, without a cosmological constant, the scale factor evolves as $a \propto t$ at late times when curvature dominates the dynamics,

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} + \frac{1}{a^2} \approx \frac{1}{a^2}\,. \tag{3.30}$$

Hence substituting $H(t) \equiv \dot{a}/a = 1/t$ into Eq. (3.24), neglecting pressure forces, gives

$$\ddot{\delta} + \frac{2\dot{\delta}}{t} = 0\,, \tag{3.31}$$

which has the general solution

$$\delta = A + B\,t^{-1}\,. \tag{3.32}$$

Again we have a decreasing mode $B\,t^{-1}$, but now the 'growing' mode does not actually increase in amplitude. Because of the low matter density, perturbations have stopped growing altogether, $\delta \to A$.

3.3.3 Matter fluctuations in a smooth relativistic background

If the Universe contains a sea of collisionless relativistic particles, for example photons or neutrinos, then they might dominate at early times, and determine the expansion rate,

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\,(\bar{\rho}_m + \bar{\rho}_R)\,, \tag{3.33}$$

with $\bar{\rho}_m \propto a^{-3}$ and $\bar{\rho}_R \propto a^{-4}$, and the growth rate of perturbations in the mass, determined by

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}_m\delta\,. \tag{3.34}$$

Here we assumed the total density to be $\rho = \bar{\rho}_m(1+\delta) + \bar{\rho}_R$, *i.e.* there are no perturbations in the radiation. The second Friedmann equation reads in this case

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) = -\frac{4\pi G}{3}(\rho_m + 2\rho_R)\,. \tag{3.35}$$

since $p_m \approx 0$ and $p_R = \rho_R/3$. Combined with the equation for the Hubble constant, we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_R(2+\eta) = -\frac{1}{2}\,\frac{\eta+2}{\eta+1}\,H^2\,, \tag{3.36}$$

where $\eta = \bar{\rho}_m/\bar{\rho}_R = \eta_0 a$. Using η as the new time variable, in which case $\dot{\eta} = \eta H$, we obtain

$$\begin{aligned} \dot{\delta} &= \delta'\,\eta H \\ \ddot{\delta} &= \delta''\,(\eta H)^2 + \delta'\,\eta_0\ddot{a} = \delta''\,(\eta H)^2 - \delta'\,\frac{1}{2}\,\frac{\eta+2}{\eta+1}\,\eta H^2\,. \end{aligned} \tag{3.37}$$

Here, δ' is $\partial\delta/\partial\eta$ and $\delta'' = \partial^2\delta/\partial\eta^2$. Therefore the perturbation equation for δ becomes

$$\frac{d^2\delta}{d\eta^2} + \frac{(2+3\eta)}{2\eta(1+\eta)}\frac{d\delta}{d\eta} = \frac{3}{2}\frac{\delta}{\eta(1+\eta)}\,. \tag{3.38}$$

The growing mode is now

$$\delta = 1 + 3\eta/2\,. \tag{3.39}$$

Clearly, the fluctuation cannot grow, $\delta \approx$ constant, until η starts to become $\eta = \bar{\rho}_m/\bar{\rho}_R \geq 1$, *i.e.*, until matter starts to dominate. When the Universe is radiation dominated, matter fluctuations cannot grow: they are tied to radiation.

3.4 Fourier decomposition of the density field

The statistics of the density field are often assumed to be Gaussian. This may follow from the central-limit theorem, or from inflation. In any case, the fluctuations *measured* in the CMB appear to be Gaussian to high level of accuracy. Here we discuss in more detail the growth of perturbations in a Gaussian density field.

3.4.1 Power-spectrum and correlation functions

We can characterise the density field by its Fourier transform,

$$\delta(\mathbf{x},t) = \sum \delta(\mathbf{k},t)\exp(i(\mathbf{k}\cdot\mathbf{x})) = \frac{V}{(2\pi)^3}\int \delta(\mathbf{k},t)\,\exp(i(\mathbf{k}\cdot\mathbf{x}))\,d\mathbf{k}\,, \tag{3.40}$$

where the density $\rho(\mathbf{x},t) = \bar{\rho}(t)\,(1+\delta(\mathbf{x},t))$, and V is a sufficiently large volume in which we approximate the Universe as periodic. Since $\delta(\mathbf{x})$ is real and has mean zero, we have $\langle\delta(\mathbf{k},t)\rangle = 0$ and $\delta(\mathbf{k},t) = \delta(-\mathbf{k},t)^\dagger$.

Since the perturbation equation Eq. (3.24) is linear, each Fourier mode $\delta(\mathbf{k},t)$ satisfies the linear growth equation separately, *i.e.*, each mode grows independently of all the others, as long as $\delta \ll 1$.

The statistical properties of the density field are determined by the infinite set of correlation functions,

$$
\begin{aligned}
\langle \delta(\mathbf{x}_1)\delta(\mathbf{x}_2) \rangle &= \xi_2 \\
\langle \delta(\mathbf{x}_1)\delta(\mathbf{x}_2)\delta(\mathbf{x}_3) \rangle &= \xi_3 \\
\langle \delta(\mathbf{x}_1)\delta(\mathbf{x}_2)\delta(\mathbf{x}_3)\delta(\mathbf{x}_4) \rangle &= \xi_4 \\
\langle \delta(\mathbf{x}_1)\delta(\mathbf{x}_2)\delta(\mathbf{x}_3)\cdots\delta(\mathbf{x}_N) \rangle &= \xi_N \,.
\end{aligned}
\tag{3.41}
$$

Since the Universe is homogeneous and isotropic, these correlation functions should only depend on relative distance, *e.g.*, $\xi_2 = \xi_2(|\mathbf{x}_1 - \mathbf{x}_2|)$ for the two-point correlation function ξ_2.

What does this mean? Consider for example the two-point density correlation function, $\xi_2(|\boldsymbol{\Delta}|) = \langle \rho(\mathbf{r})\rho(\mathbf{r}+\boldsymbol{\Delta})\rangle$. Clearly, for $\boldsymbol{\Delta} = 0$, $\xi_2(0) = \langle \rho^2 \rangle$, whereas for very large Δ, you expect the density at $\mathbf{r}$ and at $\mathbf{r}+\boldsymbol{\Delta}$ to be uncorrelated, hence $\xi_2(|\boldsymbol{\Delta}| \to \infty) \to \langle \rho \rangle^2$. In the intermediate regime, ξ_2 describes to what extent the density at two points separated by $\boldsymbol{\Delta}$ are correlated.

Inserting the Fourier decomposition into ξ_2 yields

$$
\begin{aligned}
\xi_2(\Delta) &= \frac{1}{V}\int \delta(\mathbf{x})\delta(\mathbf{x}+\boldsymbol{\Delta})\,d\mathbf{x} \\
&= \frac{1}{V}\sum_{\mathbf{k},\mathbf{q}} \int \delta(\mathbf{k})\, \exp(i\mathbf{k}\cdot\mathbf{x})\,\delta(\mathbf{q})\, \exp(i\mathbf{q}(\mathbf{x}+\boldsymbol{\Delta}))\,d\mathbf{x} \\
&= \sum_{\mathbf{k}} |\delta(\mathbf{k})|^2\, \exp(-i\mathbf{k}\cdot\boldsymbol{\Delta}) \\
&= \frac{V}{(2\pi)^3}\int |\delta(\mathbf{k})|^2\, \exp(-i\mathbf{k}\cdot\boldsymbol{\Delta})\,d\mathbf{k}\,,
\end{aligned}
\tag{3.42}
$$

where we used the fact that

$$
\frac{1}{V}\int \exp(i\mathbf{k}\cdot\mathbf{x})\,d\mathbf{x} = \delta^D(\mathbf{k})\,, \tag{3.43}
$$

where δ^D is Dirac's delta function. The function $P(k) \equiv \langle |\delta(\mathbf{k})|^2\rangle$ is called the *power spectrum* of the density field, the $\langle\rangle$ denote an ensemble average. Note that because of isotropy, $P(k) = P(|\mathbf{k}|)$. The two-point correlation function is the Fourier transform of the power-spectrum.

3.4.2 Gaussian fields

An important class of density fields is one where $\delta(\mathbf{k})$ are Gaussian variables, *i.e.* where the real and imaginary parts of δ are independently Gaussian distributed with zero mean and dispersion $P(k)/2$, such that $\langle|\delta|^2\rangle = \langle|\delta_R|^2\rangle + \langle|\delta_I|^2\rangle = P$. Such a density field is called a **Gaussian field**. By writing the complex amplitude as the sum of its real and imaginary parts, $\delta = \delta_R + i\delta_I$, and introducing the amplitude $A = (\delta_R^2 + \delta_I^2)^{1/2}$ and phase angle θ, $\tan\theta = \delta_I/\delta_R$, we can find the probability distribution for A and θ from

$$\begin{aligned} P(\delta_R)d\delta_R\, P(\delta_I)d\delta_I &= \frac{1}{2\pi P/2}\,\exp(-(\delta_R^2+\delta_I^2)/P)\, d\delta_R\, d\delta_I \\ &= \frac{1}{2\pi P/2}\, A\,\exp(-A^2/P)\, dA\, d\theta\,, \end{aligned} \tag{3.44}$$

from which it becomes clear that the amplitude of the Fourier modes are distributed $\propto A\exp(-A^2/P)$, and the phases θ are random in $[0, 2\pi[$.

If the Fourier modes are Gaussian, then the real-space density field is a sum of Gaussian fields, hence also a Gaussian field, with probability

$$p(\delta)d\delta = \frac{1}{(2\pi\sigma^2)^{1/2}}\,\exp(-\delta^2/2\sigma^2)\, d\delta\,. \tag{3.45}$$

Since $\delta > -1$, this can only work when the dispersion $\sigma \ll 1$, that is, the field can only be Gaussian for very small perturbations, and as the perturbations grow, the statistics of the field will always start to deviate from a Gaussian.

A Gaussian field has the important property that all the higher order correlation functions can be computed in terms of the power-spectrum, or alternatively from the two-point correlation function. So, *if* the cosmological density field were Gaussian initially, then its statistics are fully determined by the power-spectrum $P(k)$. Fluctuations generated during inflation are likely, but not necessarily, Gaussian.

As fluctuations grow, the statistics of δ will become non-Gaussian, in particular, the distribution of the galaxy density field today is non-Gaussian. However, for the CMB radiation, the fluctuations are very small, and are very accurately Gaussian, although there is a large army of cosmologists searching for deviations from Gaussianity.

3.4.3 Mass fluctuations

The Fourier decomposition can also be used to estimate the mass-variance in spheres of a given size, V_W. The mean mass in such a volume is $\bar{M} = \bar{\rho}\, V_W$. To find the level of fluctuation around the mean value, $\langle(\delta M/M)^2\rangle$, define a filter function W such that $\int W(\mathbf{x})\, d\mathbf{x} = V_W$. When this filter function is positioned at position $\mathbf{x}$ the mass contained within it is

$$\begin{aligned}
M(\mathbf{x}) &= \int \rho(\mathbf{x}')\, W(\mathbf{x}-\mathbf{x}')\, d\mathbf{x}' \\
&= \int \bar{\rho}(1+\delta(\mathbf{x}'))\, W(\mathbf{x}-\mathbf{x}')\, d\mathbf{x}' \\
&= \bar{\rho} \int W(\mathbf{x}-\mathbf{x}')\, d\mathbf{x}' + \bar{\rho} \int \sum_{\mathbf{k},\mathbf{q}} \delta(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{x}')\, W(\mathbf{q})\, \exp(i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}'))\, d\mathbf{x}'
\end{aligned}$$

The integral over $\mathbf{x}'$ is $\int \exp(i(\mathbf{k}-\mathbf{q})\cdot\mathbf{x}')\, d\mathbf{x}' = V\delta^D(\mathbf{k}-\mathbf{q})$, hence

$$M(\mathbf{x}) = \bar{M}\left[1 + \frac{V}{V_W} \sum_{\mathbf{k}} \delta(\mathbf{k})\, W(\mathbf{k})\, \exp(i\mathbf{k}\cdot\mathbf{x})\right]. \tag{3.46}$$

Consequently

$$\begin{aligned}
\langle(\frac{M(\mathbf{x})-\bar{M}}{\bar{M}})^2\rangle &= \frac{1}{V}\int (\frac{M(\mathbf{x})-\bar{M}}{\bar{M}})^2\, d\mathbf{x} \\
&= \frac{V}{V_W^2} \int \sum_{\mathbf{k},\mathbf{q}} \delta(\mathbf{k})\, W(\mathbf{k})\, \exp(i\mathbf{k}\cdot\mathbf{x})\delta(\mathbf{q})\, W(\mathbf{q})\, \exp(i\mathbf{q}\cdot\mathbf{x})\, d\mathbf{x} \\
&= \frac{V^2}{V_W^2} \sum_{\mathbf{k}} |\delta(\mathbf{k})|^2\, |W(\mathbf{k})|^2 \\
&= \frac{V^3}{(2\pi)^3\, V_W^2} \int |\delta(\mathbf{k})|^2\, |W(\mathbf{k})|^2\, d\mathbf{k} \\
&= \frac{1}{V_W^2} \int_0^\infty \frac{dk}{k}\, \frac{k^3\, |\delta(\mathbf{k})|^2\, V^3}{2\pi^2}\, |W(\mathbf{k})|^2 .
\end{aligned} \tag{3.47}$$

This shows that

$$\Delta^2(k) \equiv \frac{k^3\, |\delta(\mathbf{k})|^2\, V^3}{2\pi^2}\,, \tag{3.48}$$

is a measure of how much fluctuations on scale k contribute to the mass fluctutations, per unit logarithmic interval in k. Note in passing that

$$V_W = \int W(\mathbf{x})\, d\mathbf{x} = \sum_{\mathbf{k}} W(\mathbf{k}) \int \exp(i\mathbf{k}\cdot\mathbf{x})\, d\mathbf{x} = V\, W(\mathbf{k}=0)\,, \qquad (3.49)$$

hence $W(\mathbf{k}=0) = V_W/V$.

A special type of filter is one which operates directly in Fourier space, such that for example

$$\frac{\delta M}{\bar{M}} = \sum_{k<k_m} \delta(\mathbf{k})\, \exp(i\mathbf{k}\cdot\mathbf{x})\,, \qquad (3.50)$$

where $\bar{M} \propto 1/k_m^3$. Such a filter only allows modes with sufficiently small k or sufficiently long wavelength $\lambda > \lambda_{\min} 2\pi/k_m \sim M^{1/3}$ to contribute to the fluctuations in mass. For such a filter,

$$\begin{aligned} \langle(\frac{\delta M}{\bar{M}})^2\rangle &= \frac{1}{V} \sum_{k<k_m, q<k_m} \int \delta(\mathbf{k})\delta(\mathbf{q})\, \exp(i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x})\, d\mathbf{x} \\ &= \sum_{k<k_m} |\delta(\mathbf{k})|^2 \\ &= \frac{V}{2\pi^2} \int_0^{k_m} k^2\, P(k) dk\,. \end{aligned} \qquad (3.51)$$

If we know the power-spectrum at a given time t_0, then we can use our linear perturbation equations to compute the PS at any later time t, $P(k,t) = T^2(k,t)\, P(k,t_0)$. Since the Fourier modes evolve independently, the function $T(k,t)$ does not depend on the actual shape $P(k,t_0)$. It is called the transfer function. Since $P(k) = \langle|\delta(k)|^2\rangle$, T also specifies how δ grows, $\delta(k,t) = T(k,t)\delta(k,t_0)$. If the primordial field is Gaussian, then its statistics is completely determined by $P(k,t) = T^2(k,t)P(k,t_0)$, and so $T(k,t)$ and $P(k,t_0)$ together provide a complete description of how structure evolves in the linear regime.

3.4.4 The Transfer function

Super-horizon growth

So far we have compute the growth rate in the Newtonian approximation, so our results may be correct for wavelengths $\lambda = 2\pi/k \ll ct$, where t is the age of the Universe[3]. We have seen that these growth rates depend on the properties of the cosmological model, in particular, the growth rate differs in the radiation vs matter dominated Universe. But a particular mode λ will also at some early time be larger than the horizon. What happens in that case? We will follow a simplified Newtonian procedure which happens to give the right answer.

Consider a small perturbations in a spatially flat, $k = 0$, Friedmann model[4]. For the unperturbed model, the Friedmann equation is

$$(\frac{\dot{a}}{a})^2 = H^2 = \frac{8\pi G}{3}\,\rho_0\,. \tag{3.52}$$

Now consider as small perturbation another such model, with the same expansion rate, but where the density ρ_1 is a little bit higher, and therefore this corresponds to a closed Universe,

$$H^2 = \frac{8\pi G}{3}\,\rho_1 - \frac{\kappa}{a^2}\,, \tag{3.53}$$

where the curvature κ (not to be confused with the wave-number k) is positive. We shall always compare these models when they have the same Hubble constant. Comparing these two models, we find a relation between the density contrast, $\delta = (\rho_1 - \rho_0)/\rho_0$ and the curvature κ,

$$\delta \equiv \frac{\rho_1 - \rho_0}{\rho_0} = \frac{\kappa/a^2}{8\pi G\rho_0/3}\,. \tag{3.54}$$

As long as δ is small, the scale factors a for both models should be very similar, therefore we have found that δ will grow as $\delta \propto 1/\rho_0 a^2 \propto a$ in a matter-dominated model ($\rho_0 \propto a^{-3}$) and as $\delta \propto a^2$ in a radiation dominated

[3]In fact, a full general-relativistic treatment results in the same growth-rates as we have obtained.

[4]Recall that this will always be a good approximation at sufficiently early times.

Universe ($\rho_0 \propto a^{-4}$), hence

$$\begin{aligned} \delta &\propto a \;\; \text{matter dominated} && (3.55) \\ &\propto a^2 \;\; \text{radiation dominated}\,. && (3.56) \end{aligned}$$

It is clear from this derivation that a density perturbation can be seen as a change in the geometry of the background model. We have found that density perturbations will also grow when outside the Horizon. This may seem counterintuitive, since the perturbation region is not actually causally connected. One way to think of it is in terms of kinematic growth, as opposed to dynamic growth.

The transfer function in a Cold Dark Matter model

We are now in a position to infer the general shape of the transfer function $T(k)$. Consider perturbations with large λ that enter the horizon late, when the Universe already has become matter dominated. All these waves will experience the same growth rate $\delta \propto a$, both when outside the Horizon, Eq. (3.55) and when inside the Horizon, Eq. (3.29), hence $T(k = 2\pi/\lambda)$ is constant and the shape of the primordial spectrum is conserved.

However, now consider a smaller wavelength perturbation, one that enters the Horizon when the Universe is still radiation dominated. Equation (3.39) shows that the amplitude of the perturbation remains constant once it has entered the Horizon, and can only start to grow when the Universe becomes matter dominated. Outside the horizon, they grew $\propto a^2$, but they stopped growing after entering the Horizon. Now consider two such perturbations, with co-moving wavelengths λ_1 and $\lambda_2 < \lambda_1$. The ratio of scale-factors a_1 and a_2 when these perturbation enter the Horizon will scale as $a_1/a_2 = \lambda_1/\lambda_2$, and hence their growth rate as $(a_1/a_2)^2 = (k_2/k_1)^2$, so $T(k) \propto 1/k^2$: the smaller the wavelength, the earlier it enters the Horizon, and the sooner its growth gets quenched due to the radiation.

Combining these considerations, we expect that

$$\begin{aligned} T(k) &\propto 1 \;\; \text{for } k \ll k_{\rm eq} \\ &\propto 1/k^2 \;\; \text{for } k \gg k_{\rm eq}\,, && (3.57) \end{aligned}$$

where $\lambda_{\rm eq} = 2\pi/k_{\rm eq}$ is the co-moving Horizon size at matter-radiation equality,

$$\lambda_{\rm eq} \approx 10\,(\Omega_m h^2)^{-1}{\rm Mpc}. \tag{3.58}$$

In practise, the roll-over from $T \approx 1$ to $T \approx 1/k^2$ is rather gradual.

An often used approximation is

$$T(k) = [1 + ((ak) + (bk)^{3/2} + (ck)^2)^\nu]^{-1/\nu}\,, \tag{3.59}$$

where $a = 6.4\,(\Omega_m h^2)^{-1}{\rm Mpc}$, $b = 3.0\,(\Omega_m h^2)^{-1}{\rm Mpc}$, $c = 1.7\,(\Omega_m h^2)^{-1}{\rm Mpc}$, and $\nu = 1.13$, plotted in Figure 3.1.

Some small remarks. (1) It is customary to normalise the transfer function such that $T = 1$ on large scales. It does not mean that such large wavelength perturbations do not grow: they do. (2) This analysis only applies to dark matter and radiation. But if we want to make observations, we would like to know what the baryons, that we see in galaxies, do. There is an extra complication here: even when the Universe is matter dominated, baryons are still tightly coupled to the radiation when the plasma is ionised, due to Thomson scattering. Therefore, before recombination, whereas the dark matter perturbations are already happily growing, the baryon perturbations are tied to the radiation and grow much less fast. Once the plasma recombines, they will quickly catch-up with the dark matter. On smaller scales, pressure forces may prevent perturbations from growing, leading to oscillations in $T(k)$ that correspond to sound waves. But the general picture is that the total growth of waves with $\lambda \ll \lambda_{\rm eq}$ is suppressed with respect to larger-scale perturbations, because their growth is temporarily halted as they enter the Horizon during radiation domination. This leads to $T(k)$ bending gradually from $T(k) \approx 1$ for $k \ll k_{\rm eq}$ and $T(k) \propto k^{-2}$ on small scales.

Finally, a proper calculation of $T(k,t)$ requires that we take all types of matter and radiation into account, and follow the growth of each mode outside and inside the Horizon. Such calculations are now routinely done numerically. For example the CMBFAST code developed by Seljak and Zaldariagga will compute $T(k,t)$ for any set of cosmological parameters.

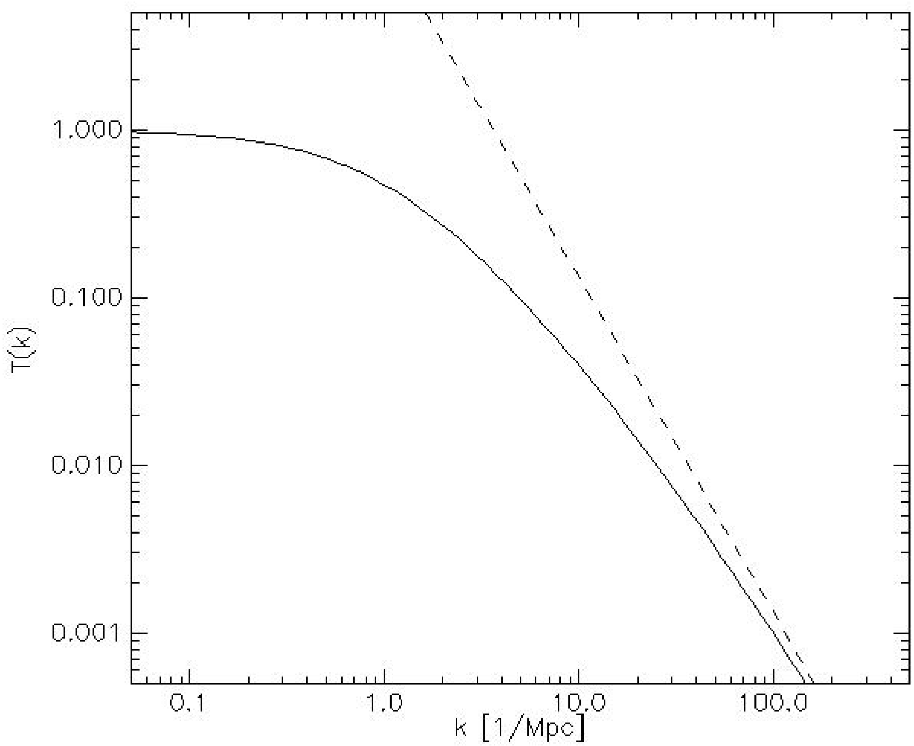

Figure 3.1: Transfer function of Eq. (3.59) in a Cold Dark Matter model, which curves slowly from $T \sim 1$ on large scales, to $T \propto 1/k^2$ on small scales. The dashed line is $T(k) \propto 1/k^2$, which corresponds to the small scale limit $k \to \infty$.

3.4.5 The primordial power-spectrum

Although we have computed how perturbations grow due to gravity, we have not yet been able to predict the ultimate origin of the density fluctuations. Harrison and independently Zel'dovich gave a symmetry argument, which we will reproduce below, and suggested that $P(k, t \to 0) \propto k^n$, with $n = 1$, called a scale-invariant or Harrison-Zel'dovich spectrum. They do not provide a mechanism to compute the amplitude of this spectrum (*i.e.* the proportionality constant).

In inflation, the ultimate origin of fluctuations are quantum mechanical. On a sub-atomic scale, quantum fluctuations will naturally produce small fluctuations. During inflation, these quantum fluctuations are 'inflated' to macroscopic scales. The predicted spectrum is close to scale-invariant, $n = 1$. We have no good way yet to compute the expected amplitude of such inflationary perturbations.

There is in fact no a priori reason that the fluctuations should have once been small. In theories with topological defects, structure is generated by defects, which you can think of as begin similar to defects in the structure of crystals. Such defects are non-perturbatively, *i.e.* they were always of finite amplitude.

The measured CMB fluctuations appear to require scale-invariant adiabatic perturbations as predicted by the simplest inflationary models, and at the moment other mechanisms for generating perturbations have gone out of fashion.

Let us write the primordial spectrum as $P(k, t \to 0) \equiv P(k)$. The argument of Harrison and Zel'dovich is that we demand that there is no physical scale associated with $P(k)$. This suggests that $P(k) = k^n$, for some value of n. Consider a sphere of co-moving radius R, and typical mass $M \propto R^3$. With a spectrum of fluctuations $P(k) \propto k^n$, the fluctuations in mass will be (cfr.Eq. (3.51))

$$\langle (\delta M/M)^2 \rangle = \int_0^{1/R} P(k)\, k^2\, dk \sim M^{-(n+3)/3} \,. \tag{3.60}$$

The corresponding potential fluctuations are of order

$$\delta\Phi \sim \frac{\delta M}{R} \sim M^{-(n+3)/6}\, M^{2/3} \sim M^{(1-n)/6}\,. \tag{3.61}$$

For $n = 1$, these are independent of scale. Another way in which $n = 1$ is scale-invariant, is to consider the amplitude of the fluctuations at the moment they enter the Horizon. The variance of density fluctuations in a patch of co-moving size is x_f is

$$\langle(\delta\rho/\rho)^2\rangle = t^2 \int_0^{1/x_f} P(k)\, k^2\, dk \sim t^2\, x_f^{-(n+3)}\,, \tag{3.62}$$

where we have used that the fluctuation $\delta \propto a^2 \propto t$ outside the Horizon in the radiation era. The fluctuation enters the Horizon when $a\, x_f = ct$ or $x_f \propto t/a \propto t^{1/2}$ since $a = t^{1/2}$ in the radiation dominated era. So when the perturbation enters the Horizon, its amplitude is $\propto x_f^4\, x_f^{-(n+3)} = x_f^{1-n}$, and is independent of x_f when $n = 1$.

3.5 Hierarchical growth of structure

So far we have assumed that the fluctuation amplitude $\delta << 1$. As δ grows, our analysis eventually breaks down, but we can already guess what will happen: a region with $\delta \sim 1$ will collapse onto itself, and in effect detach itself from the Hubble expansion. Presumably this is what happens during galaxy formation. We will treat this in more detail below, but less us investigate how this process depends on scale.

Let the power-spectrum at late times $P(k,t) = T(k,t)^2\, P(k,t_0) \propto k^m$, and we expect $m \sim 1$ on large scales, and $m \sim -3$ on small scales. In that case, we can use our reasoning that lead to Eq. (3.60) to find that the rms mass fluctuations as function of scale are

$$\langle(\delta M/M)^2\rangle = \int_0^{1/R} P(k)\, k^2\, dk \sim M^{-(m+3)/3}\,. \tag{3.63}$$

If $m > -3$, then the fluctuation amplitude decreases with increasing scale, and hence small objects will first become non-linear, $\delta \sim 1$, and hence they will collapse before more massive ones. This phenomena is called *hierarchical growth*, were more massive objects form from the merging of smaller structures. Given its value for m, this is the case for the CDM model.

3.5.1 Statistics of density peaks

The statistics of Gaussian fields, such as for example the number density of local maxima, have been studied in great detail, see Bardeen et al., ApJS 304, 15, 1985[5] For simplicity, consider a 1D Gaussian field, smoothed on some scale R,

$$\delta(x, R) = \int \delta(k) W(kR) \exp(ikx)\, dk\,., \tag{3.64}$$

where W is a smoothing function.[6] If δ is a Gaussian field, then also $\delta' = d\delta/dx$ and all other derivatives are Gaussian fields, with dispersions

$$\begin{aligned}
\langle \delta^2 \rangle &= \int P(k)\, W^2(kR)\, dk \equiv \sigma_0^2 \\
\langle \delta'^2 \rangle &= \int k^2\, P(k)\, W^2(kR)\, dk \equiv \sigma_1^2 \\
\langle \delta''^2 \rangle &= \int k^4\, P(k)\, W^2(kR)\, dk \equiv \sigma_2^2\,.
\end{aligned} \tag{3.65}$$

What we are after is the number density of local maxima, where $\delta' = 0$ and $\delta'' < 0$. Now, if we ask in how many points $\delta' = 0$, then we always find 0: it is a set of measure zero. What we mean is not that $\delta' = 0$, but $\delta' = 0$ within some some small volume dx around the peak. Then, if the peak is at x_p, close to the peak

$$\delta'(x) = \delta''(x_p)\,(x - x_p) \tag{3.66}$$

hence

$$\frac{\delta'(x)}{\delta''(x_p)} = x - x_p\,, \tag{3.67}$$

so that

$$\delta^D[(x - x_p)] = \delta^D[\delta'(x)/\delta''(x_p)] = \delta''(x_p)\,\delta^D[\delta'(x)]\,, \tag{3.68}$$

[5] In April 2004, this paper has been references 1370 times.

[6] Recall that the Fourier transform of a convolution of two functions, $f * g$, is the product of the Fourier transforms, $\widehat{f * g} = \hat{f}\,\hat{g}$.

where δ^D is again the Dirac delta function. The number density $n_{\rm pk}(\delta)$ of peaks of height δ then follows from

$$\begin{aligned} n_{\rm pk}(\delta) &= \langle \delta^D[(x - x_p)] \rangle \\ &= \langle \delta''(x_p)\, \delta^D[\delta'(x)] \rangle \\ &= \int P(\delta, \delta' = 0, \delta'')\, |\delta''| d\delta\, d\delta''\,. \end{aligned} \tag{3.69}$$

The importance of peaks is that it helps to understand why unusual objects, for example massive galaxy clusters, seem to occur close to each other. For high peaks, Kaiser shows that the correlation function

$$\langle \delta_{\rm pk}(\delta > \nu\sigma_0, \mathbf{r}) \delta_{\rm pk}(\delta > \nu\sigma_0, \mathbf{r} + \mathbf{\Delta}) \rangle = \frac{\nu^2}{\sigma_0^2} \langle \delta(\mathbf{r}) \delta(\mathbf{r} + \mathbf{\Delta}) \rangle\,. \tag{3.70}$$

3.6 Perturbations in the Cosmic Micro-wave Background

We have just computed the expected power-spectrum of fluctuations, $P(k) = T^2(k)\, k \propto k$ on large scales, and $\propto k^{-3}$ on small scales, where 'large' means with co-moving wavelength larger than the Horizon at matter-radiation equality. These perturbations will induce perturbations in the CMB temperature on the sky.

To describe this, expand the temperature fluctuations on the celestial sphere in spherical harmonics,

$$\frac{\delta T(\theta, \phi)}{T} = \sum_{l,m} a_{lm}\, Y_{lm}(\theta, \phi)\,. \tag{3.71}$$

Here, (θ, ϕ) denotes a given direction in the sky, and Y_{lm} are the usual spherical harmonics. If the fluctuations are Gaussian, then they are completely characterised by their power-spectrum

$$C_l = \langle |a_{lm}|^2 \rangle\,, \tag{3.72}$$

which also defines the temperature auto-correlation function,

$$C(\theta) = \langle \frac{\delta T(\mathbf{n}_1)}{T} \frac{\delta T(\mathbf{n}_2)}{T} \rangle = \frac{1}{4\pi} \sum_l (2l+1)\, C_l P_l(\cos(\theta))\,, \tag{3.73}$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are two directions on the sky, separated by the angle θ.

How do we expect the power-spectrum to look like? Let us begin by computing the angular size $\theta_{\rm rec}$ of the Horizon at recombination, $z = z_{\rm rec} \sim 1100$. Using Eq. (2.51) gives an angular-size diameter distance to the last-scattering surface (LSS) of

$$d_A \approx \frac{3t_0}{z_{\rm rec}}\,. \tag{3.74}$$

The physical size of the Horizon, $a_{\rm rec}\, r_e$, at recombination follows from the reasoning that lead to Eq. (2.50), but assuming radiation domination, $a(t) = a_{\rm rec}\,(t/t_{\rm rec})^{1/2}$, in which case $a_{\rm rec}\, r_e = 2t_{\rm rec}$. Finally, $a_{\rm rec} = (t/t_0)^{2/3}$. Combining all this, gives

$$\theta_{\rm rec} = \frac{a_{\rm rec}\, r_e}{d_A} = \frac{2}{3} \frac{1}{(1+z_{\rm rec})^{1/2}} \approx 0.02\,. \tag{3.75}$$

so $\theta_{\rm rec} \approx 1.15^\circ$ (We have assumed an Einstein-de Sitter Universe all the way, the more accurate value $0.87^\circ\, \Omega_m^{1/2}$ depends on the matter density.)

This will be an important scale for the CMB fluctuations, because perturbations on larger scale were outside of the Horizon at decoupling, and hence were not susceptible to any causal processes.

There are five distinct physical effects that give rise to CMB fluctuations, they are

1. our peculiar velocity wrt the CMB rest frame
2. fluctuations in the gravitational potential at the LSS
3. fluctuations in the intrinsic radiation field itself
4. peculiar velocities of the LSS
5. changes to the CMB BB spectrum due to effects on the photons as they travel from the LSS to us

The first arises because the Milky Way is not a fundamental observer. Its motion introduces a dipole in the CMB, which cannot be distinguished from any intrinsic dipole. The second effect, called the Sachs-Wolf effect, arises because photons redshift due to gravitational redshift, as they climb out of a potential well, and so these fluctuations are a measure of fluctuations in the gravitational potentential due to large-scale perturbations. It is the dominant effect on large scales, $\theta > \theta_{\rm rec}$. The last three effects dominate on smaller scales.

The COBE satellite[7] was the first to measure temperature fluctuations in the CMB (Fig.3.2). Its antenna had an angular resolution of around 7 degrees, so was only able to measure super-horizon scales! The top panel in the Figure shows the strong dipole contribution, which has been subtracted in the lower two panels. The regions of slightly hotter or slightly coolder show-up as red and blue regions. The size of the regions is mostly determined by the resolution of the COBE, not the intrinsic size of the fluctuations. The large red band running through the centre of the middle panel is micro-wave emission from dust in the Milky Way disk, it is not due to CMB photons. The regions contaminated by this 'foreground' emission is masked in the botton panel. CMB experiments such as COBE measure the CMB sky in a range of wavelengths, in order to be able to distinguish the true CMB signal from foreground emission such as dust in the Milky Way and other galaxies. The typical rms temperature variation in the CMB sky is only $\langle(\delta T/T)^2\rangle^{1/2} \approx 10^{-5}$.

The WMAP satellite[8] dramatically improved the resolution of COBE, see Fig. 3.3) and produced the beautiful spectrum shown in Fig.3.4. Note the tremendously small error bars!

The red line is a fit to the data, assuming a CDM model with a cosmological constant, where fluctuations start from a scale-invariant spectrum, as we discussed earlier. It has around 10 parameters, for example Ω_m, Ω_Λ, n, h, $\cdots$.

Note that this model describes the true 'mean' power-spectrum. The power-spectrum is defined as $C_l = \langle|a_{lm}|^2\rangle$. The angular brackets refer to a

[7]http://aether.lbl.gov/www/projects/cobe/
[8]http://lambda.gsfc.nasa.gov/product/map/

mean over many realisations. However, we can measure only one realisation: the actual CMB sky we see from our vantage point. So we would not expect the actual CMB to be represented by the mean expectation. How far an actual measurement can differ from the mean, is illustrated by the grey band. Clearly, on small scales, C_l has many constributions from multipoles with $-l \leq m \leq l$, but the low-order multipoles have many fewer contributions. Consequently the expected value of C_l is close to the mean value for large l (small scales), whereas the grey band widens at small l (large scales) where an actual single realisation may differ considerably from the mean. This phenomena is called 'cosmic variance'.

The tremendous agreement between CMB data and the theoretical expectation is a strong indication that the models we have been discussing are relevant for the actual Universe.

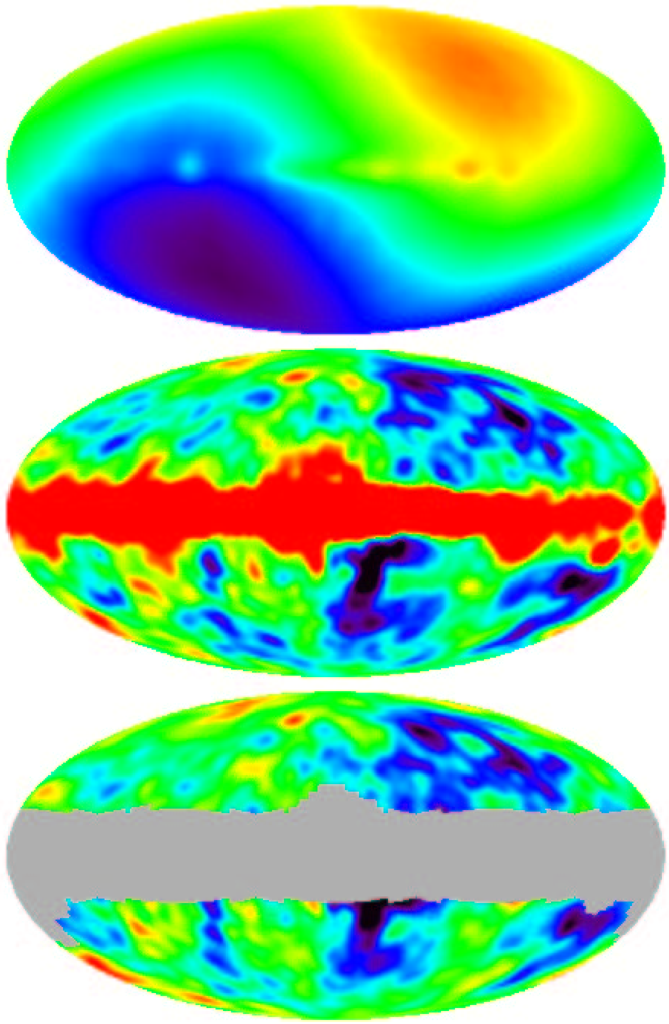

Figure 3.2: Temperature fluctuations on the CMB sky, as measured by COBE. The colour scale is a measure of the temperature of the CMB sky in that direction. The top panel is dominated by the dipole resulting from the Milky Way's motion. This dipole is subtracted in the lower panels. The disk of the Milky Way runs horizontally through the image. Clearly it is an important contaminant.

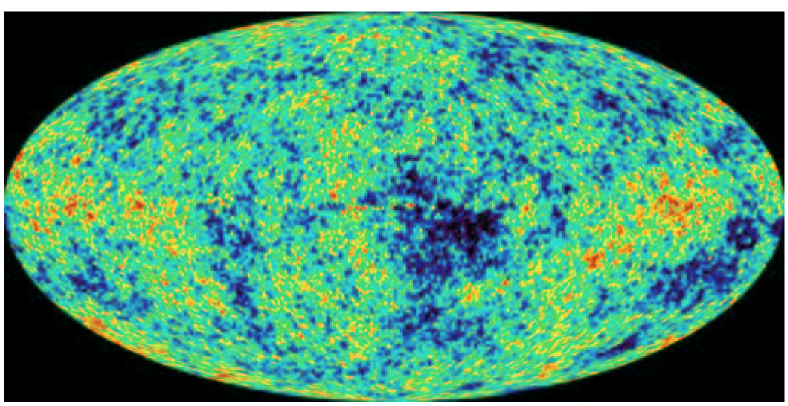

Figure 3.3: Same as Fig. 3.2, but for the WMAP satellite. Note that the large-scale features in COBE and WMAP correspond very well. WMAP has far better angular resolution, around 0.22° for the best chanel.

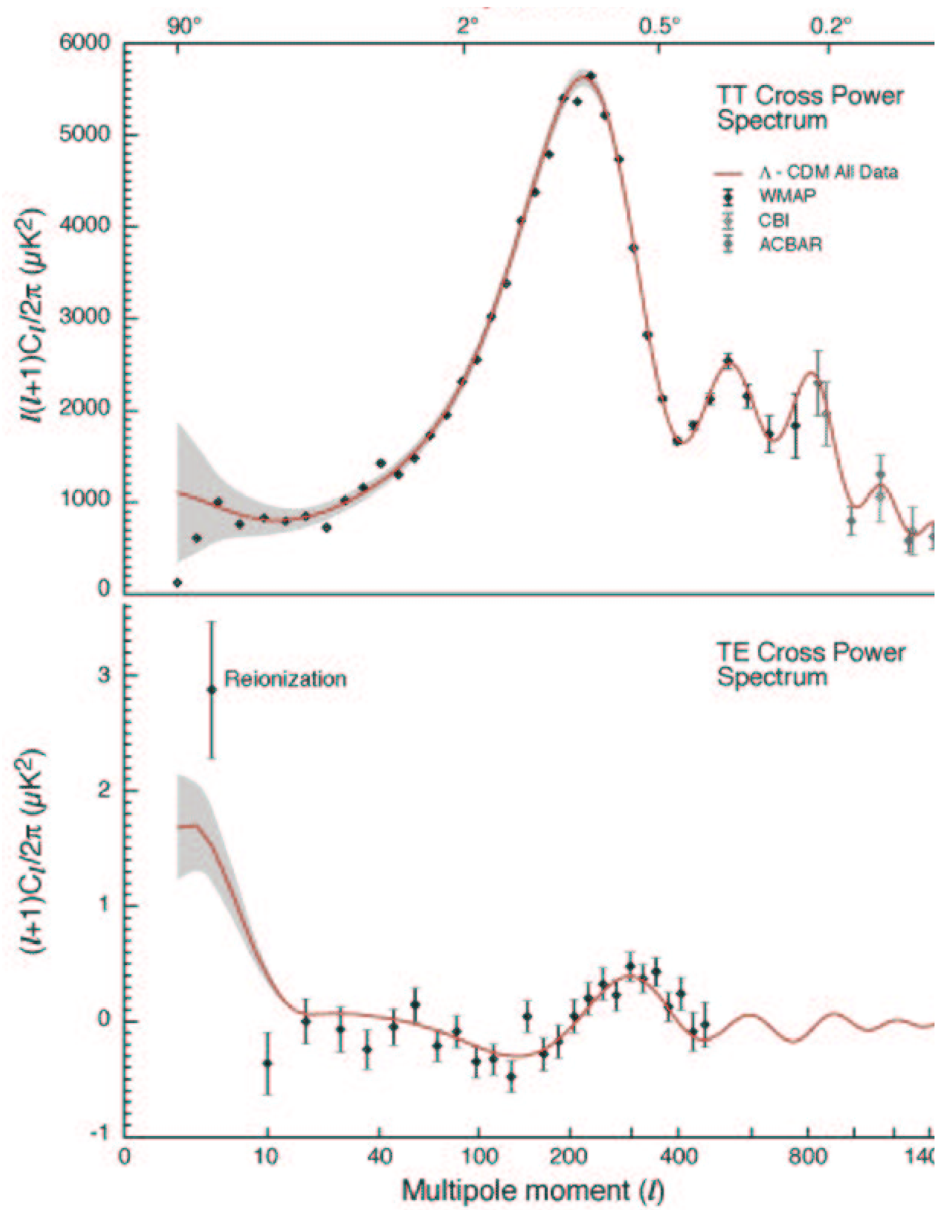

Figure 3.4: Temperature fluctuations power-spectrum as measured by the WMAP satellite (top-panel). CBI and ACBAR are balloon born CMB experiments.(see text)

Chapter 4

Non-linear growth of Cosmological Perturbations

The previous discussion focused on the linear growth of fluctuations. The density of the Milky Way disk is several hundred times the mean density, and so clearly linear growth is not sufficient to describe the formation of galaxies.

4.1 Spherical top-hat

Consider a spherical region of uniform density ρ and radius R. Newton's theorem (or Birkhoff's in general relativity) guarantees that the collapse of this object is independent of the mass-distribution around it, hence

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} \tag{4.1}$$

where $M = (4\pi/3)\,\rho R^3$ is a constant. The first integral expresses conservation of energy E,

$$\frac{1}{2}\,(\frac{dR}{dt})^2 - \frac{GM}{R} = E\,. \tag{4.2}$$

For $E < 0$, this is a bound system, with the usual parametric solution

$$\begin{aligned} \frac{R}{R_m} &= \frac{1}{2}\,(1-\cos\eta) \\ \frac{t}{t_m} &= \frac{1}{\pi}\,(\eta - \sin\eta)\,. \end{aligned} \tag{4.3}$$

From $R = 0$ at $t = 0$, the object expands, reaches a maximum size R_m at $t = t_m$ ($\eta = \pi$), and then collapse to zero size again at $t = 2t_m$ ($\eta = 2\pi$). Substituting the solution into the equation of motion, we find that

$$\begin{aligned} E &= -\frac{GM}{R_m} \\ t_m &= \frac{\pi R_m}{(-8E)^{1/2}} \,. \end{aligned} \tag{4.4}$$

The behaviour at early times, $\eta \ll 1$, is found by Taylor-expanding Eqs. (4.3),

$$\begin{aligned} \frac{R}{R_m} &= \frac{\eta^2}{4}\,(1 - \frac{\eta^2}{12} + \cdots) \\ \frac{t}{t_m} &= \frac{\eta^3}{6\pi}\,(1 - \frac{\eta^2}{20} + \cdots) \end{aligned} \tag{4.5}$$

Iteratively solving this set of equations allows us to find $R = R(t)$ as

$$R = \frac{R_m}{4}\,(\frac{6\pi t}{t_m})^{2/3}\left[1 - \frac{1}{20}\,(\frac{6\pi t}{t_m})^{2/3} + \cdots\right] \tag{4.6}$$

The second term in the [] brackets is the first-order correction to the expansion, and hence corresponds to the deviation from an Einstein-de Sitter Universe. Writing the enclosed mass in radius R as $(4\pi/3)\bar{\rho}R_0^3$ for the EdS Universe, and as $(4\pi/3)\bar{\rho}(1+\delta)R^3$ for the small perturbation on top of the EdS, we get that when the radii enclose the same mass, $R = (1 - \delta/3)R_0$. Comparing with Eq. (4.6) shows that the mean over density, with respect to an EdS Universe of the same age, is

$$\delta = \frac{3}{20}\,(\frac{6\pi t}{t_m})^{2/3}\,. \tag{4.7}$$

The non-linear collapse of the sphere to $R = 0$ occurs at $t = 2t_m$, at which time the extrapolated *linear* over-density is

$$\delta_{\rm collapse} = \delta(t = 2t_m) = \frac{3}{20}\,(12\pi)^{2/3} \approx 1.686 \equiv \delta_c\,. \tag{4.8}$$

The advantage of this result is that we can use linear perturbation theory to follow the growth of the perturbation, and decide that the full non-linear collapse will occur when the linear over density reaches 1.7. This is used in the Press-Schechter theory of the mass function of collapsed objects.

4.2 Press-Schechter mass function

In Eq. (3.47), we computed the expected mass-fluctuations on a given scale in a Gaussian field, characterised with its power-spectrum $P(k)$, as

$$\begin{aligned}
\Delta^2(\bar{M}, t) &\equiv \langle(\frac{M(\mathbf{x}, t) - \bar{M}}{\bar{M}})^2\rangle \\
&= \frac{1}{V_W^2} \int_0^\infty \frac{dk}{k} \frac{k^3\, |\delta(\mathbf{k}, t)|^2\, V^3}{2\pi^2} |W(\mathbf{k})|^2 \\
&= D^2(t) \frac{1}{V_W^2} \int_0^\infty \frac{dk}{k} \frac{k^3\, |\delta(\mathbf{k}, t_i)|^2\, V^3}{2\pi^2} |W(\mathbf{k})|^2 \\
&= D^2(t)\, \Delta^2(\bar{M}, t_i)\,. \qquad (4.9)
\end{aligned}$$

Here, $W(k)$ is the Fourier-transform of a filter function, that selects volumes of size V_W and mean mass $\bar{M}$, and $\Delta^2 = \langle(\frac{M(\mathbf{x},t)-\bar{M}}{\bar{M}})^2\rangle$ are the fluctuations in mass M around the mean enclosed mass $\bar{M} = \bar{\rho} V_W$. $\delta(\mathbf{k}, t)$ are the Fourier amplitudes of the density field, as in

$$\rho(\mathbf{x}, t) = \bar{\rho}(t) \sum_{\mathbf{k}} \delta(\mathbf{k}, t) \exp(i\mathbf{k} \cdot \mathbf{x})\,. \qquad (4.10)$$

In a given cosmological model, we know how the amplitude $\delta(\mathbf{k}, t)$ grows in time $\delta(\mathbf{k}, t) = D(t)\delta(\mathbf{k}, t_i)$, with for example $D(t) = (t/t_i)^{2/3}$ in the EdS model, and t_i some initial early time at which we know δ.

Now assume we have a Gaussian field $\delta(\mathbf{k}, t_i)$ at some early time $t = t_i$. We can use linear theory to predict the evolution of the density field, using $\delta(\mathbf{k}, t) = D(t)\, \delta(\mathbf{k}, t_i)$. This field, smoothed on the scale $\bar{M}$ is

$$\begin{aligned}
\delta(\bar{M}, \mathbf{x}, t) &\equiv \int \bar{\rho}(t)\, \delta(\mathbf{x}', t)\, W(\mathbf{x} - \mathbf{x}')\, d\mathbf{x}' \\
&= \sum \bar{\rho} V \sum_{\mathbf{k}} \delta(\mathbf{k}, t)\, W(\mathbf{k})\, \exp(-i\mathbf{k} \cdot \mathbf{x}) \\
&= \bar{\rho}\, D(t)\, \frac{V^2}{(2\pi)^3} \int \delta(\mathbf{k}, t_i)\, \exp(-i\mathbf{k} \cdot \mathbf{x})\, d\mathbf{k}\,. \qquad (4.11)
\end{aligned}$$

Since $\delta(\mathbf{k}, t_i)$ is a Gaussian field, $\delta(\bar{M}, \mathbf{x}, t)$ is a sum of Gaussian fields, hence also a Gaussian field, with mean zero and rms $\Delta(\bar{M}, t)$, hence probability distribution

$$P(\delta(\bar{M},\mathbf{x},t))d\delta(\bar{M},\mathbf{x},t) = \frac{1}{(2\pi\Delta^2(\bar{M},t))^{1/2}}\ \exp(-\delta(\bar{M},\mathbf{x},t)/2\Delta^2(\bar{M},t))\, d\delta(\bar{M},\mathbf{x},t)\,. \tag{4.12}$$

In an inspired paper, Press and Schechter (1974) suggested how to use this to compute the mass function of galaxies. The argument goes as follows.

Suppose we know the shape of the power-spectrum, $P(k,t_i) \propto k^m$, say, then we find for the mass dependence of Δ, that (cfr. Eq. (3.47))

$$\Delta(\bar{M},t) \propto D(t)\,\bar{M}^{-(m+3)/6}\,. \tag{4.13}$$

Take for example $m = -2$, so that $\Delta(\bar{M},t) = A\,D(t)/M^{1/6}$. On sufficiently large scales, such that $\Delta(\bar{M},t) \ll 1$, $\delta(\bar{M},\mathbf{x},t)$ is a Gaussian field with mean zero and very small dispersion, and almost all points still in the linear regime $\delta < 1$. Now, on smaller scales, the dispersion increases, hence some small fraction of points have $\delta \approx 1$. Comparing to our spherical top-hat model, this suggests that this fraction of space may undergo non-linear collapse and detach from the expansion of the Universe, and form a galaxy. We can also keep the filter mass constant, $\bar{M}$ = constant, but investigate the field on later and later times. The dispersion now increases because $D(t)$ increases, and so whereas the field on the large scale $\bar{M}$ was initially linear everywhere, it will eventually becomes $\delta \sim 1$ over some fraction of space, and objects with this large mass $\bar{M}$ will start to form anyway. This is the essence of the hierarchical growth of structure.

Using what we learned from the spherical collapse model, we can identify $\delta_c = 1.686$ with the critical linear density, above which the object collapses. The fraction of space for which the linear density is above this threshold, is

$$\begin{aligned} F(\bar{M},t) &= \int_{\delta_c}^{\infty} d\delta \frac{1}{(2\pi)^{1/2} D\Delta(\bar{M},t_i)}\ \exp\left[-\delta^2/2D^2\Delta^2(\bar{M},t_i)\right] \\ &= \int_{\delta_c/D\,\Delta}^{\infty} \frac{1}{(2\pi)^{1/2}}\ \exp(-x^2/2)\,dx\,, \end{aligned} \tag{4.14}$$

where $x = \delta_c/D\Delta$. Press and Schechter (1974) made the assumption that this fraction be identified with the fraction of space that has collapsed into objects of mass at least $\bar{M}$. If $n(\bar{M})$ denotes the number density of objects with mass $\bar{M}$, then, since they occupy a volume $V = \bar{M}/\bar{\rho}$, we have according to Press & Schechter that,

$$F(\bar{M}, t) = \int_{\bar{M}}^{\infty} n(\bar{M}) \, \frac{\bar{M}}{\bar{\rho}} \, d\bar{M} \,. \tag{4.15}$$

However, consider what happens for $\bar{M} \to 0$ in Eq. (4.14), in which case $\Delta \to \infty$, and the integral over x runs from 0 to ∞, hence $F(\bar{M} \geq 0, t)$=1/2: only half of the whole Universe is expected to collapse in objects of any size. This seems strange, because at sufficiently small $\bar{M}$, the density field always becomes non-linear (at least for our choice of $P(k)$), and so you'd think that the Universe will always collapse in sufficiently small objects. To overcome this problem, Press and Schechter arbitrarily multiplied Eq. (4.14) by a factor 2 to take this into account. With an extra factor of 2 in that equation, we can equate the derivative of Esq. (4.14) and (4.15) wrt $\bar{M}$, to obtain an expression for $n(\bar{M})$ as

$$-n(\bar{M}) \, \frac{\bar{M}}{\bar{\rho}} \, d\bar{M} = -2 \frac{1}{(2\pi)^{1/2}} \, \exp(-\delta_c^2/2D^2 \, \Delta^2) \frac{-\delta_c}{D \, \Delta^2} \, \frac{d\Delta(\bar{M})}{d\bar{M}} \, d\bar{M} \,, \tag{4.16}$$

which can be re-arranged as

$$n(\bar{M}, t) \, d\bar{M} = -(\frac{2}{\pi})^{1/2} \, \frac{\bar{\rho}}{\bar{M}} \, \frac{\delta_c}{D(t) \, \Delta^2(\bar{M}, t_i)} \, \frac{d\Delta(\bar{M}, t)}{d\bar{M}} \, \exp\left[-\frac{\delta_c^2}{2D^2(t) \, \Delta^2(\bar{M}, t_i)}\right] d\bar{M} \,. \tag{4.17}$$

For $P(k) \propto k^m$, $\Delta(\bar{M}) \propto \bar{M}^{-(m+3)/6}$ and

$$n(\bar{M}, t) \, d\bar{M} = (\frac{1}{2\pi})^{1/2} \, \frac{\bar{\rho}}{\bar{M}} \, (1 + \frac{m}{3}) \, (\bar{M}/M_\star(t))^{(3+m)/6} \, \exp\left[-(\bar{M}/M_\star(t))^{(3+m)/3}\right] \frac{d\bar{M}}{\bar{M}} \,, \tag{4.18}$$

where the characteristic mass $M_\star(t)$ is defined as $D(t) \, \Delta(M_\star(t), t_i) = \delta_c$.

At a given time, $M_\star(t)$ has some value, and Eq. (4.18) shows that the mass function $n(\bar{M})$ cuts-off exponentially at large masses $\bar{M} \gg M_\star(t)$, and is a power-law $n(\bar{M}) \propto \bar{M}^{(m-9)/6} \propto 1/\bar{M}^2$ for $m = -3$ (Recall that $m = -3$ on small scales.)

Now compare the mass function at two different times, $t_1 < t_2$. At t_1, the mass cut-off will be at $M_\star(t_1) < M_\star(t_2)$, that is, there will be a larger fraction of massive objects at t_2 than at t_1: some fraction of objects at t_1 has merged with other objects to form these more massive objects at t_2. With this value of the spectral index, we can expect the Universe to evolve from having very many, very low-mass objects at early times, to a growing number of more and more massive objects that form from the merging of smaller objects, at later time. In particular, an object such as the Milky Way, will have been broken-up into many smaller components, a long time ago. This is called 'hierarchical formation' of objects. The value of the mass $M\star$ now is of order of the Milky Way mass. Because of this, most of the mass currently is in MW sized objects, whereas the fraction of mass in much more massive objects, such as groups and clusters of galaxies, is exponentially suppressed.

The non-linear evolution of the Universe can also be studied with computer simulations, to be discussed below. The agreement between the mass function found from simulations, and the PS mass function, is really rather good, much better than could have been expected for such a rather simplified derivation. There is actually a better way to understand where the arbitrary factor two that we introduced comes from, although it took many years before anybody came-up with the solution to the puzzle. It is explained in Bond et al.'s excursion set approach.

4.3 Zel'dovich approximation

The perturbative equations we derived earlier require the density contrast $\delta \ll 1$. A very powerful alternative is to perform Lagrangian perturbation theory, as introduced by Zel'dovich.

Start from Eq. (3.24), neglecting pressure:

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\rho_b\delta \,. \tag{4.19}$$

Since there are no spatial derivatives in this eqation for δ, it follows that the density contrast grows as

$$\delta(\mathbf{x}, t) = D(t)\,\delta(\mathbf{x}_0)\,, \tag{4.20}$$

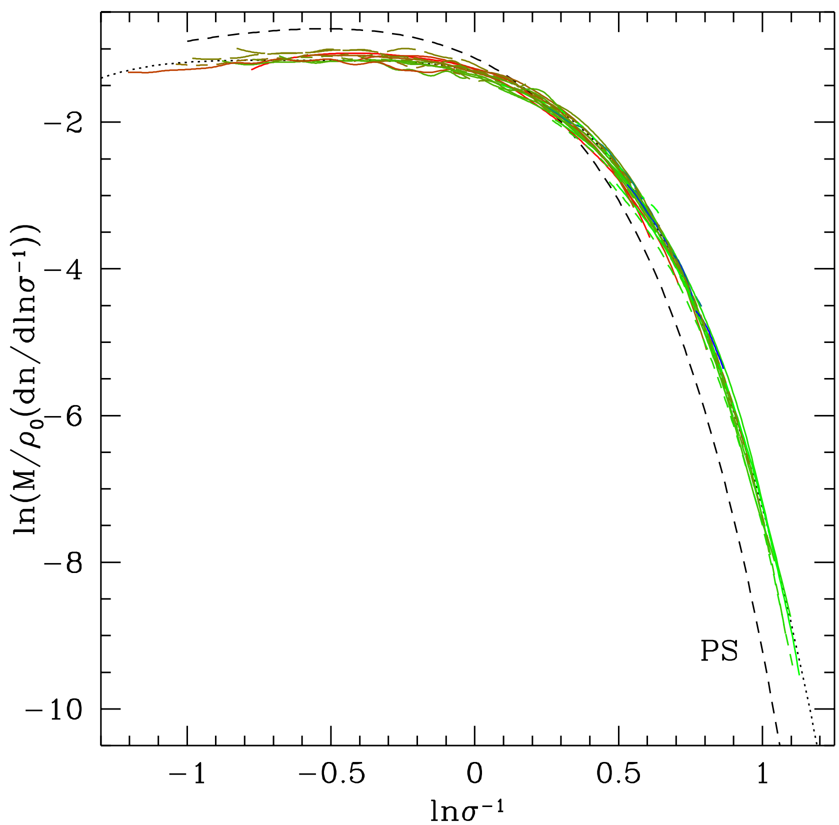

Figure 4.1: Comparison of simulated mass function (wave-lines) with the Press-Schechter mass function (dashed line, labelled PS). σ^{-1} is a measure of the mass of the object. The mass function has an almost exponential cut-off at large M, and becomes a power-law at small masses. PS mass-function also has this general shape, but under-predicts the number of massive haloes, and over-predicts the number of low mass haloes. From Jenkins et al., 2000

where D is the growth factor, and $\delta(\mathbf{x}_0)$ is the density field at some initial time. Clearly, the density field grows *self-similarly*, *i.e.* it grows $\propto D$ everywhere.

Inserting this in the Poisson equation (Eq.3.23),

$$\nabla^2\psi(\mathbf{x},t) = 4\pi G\rho_b\delta a^2\,, \tag{4.21}$$

shows that

$$\psi(\mathbf{x},t) = \frac{D}{a}\,\psi(\mathbf{x}_0)\,. \tag{4.22}$$

In the Einstein-de Sitter case, $D \propto a$, and ψ is independent of time. Finally, using the linear form of Euler's equation, (Eq.3.23),

$$\ddot{\mathbf{x}} + 2H\dot{\mathbf{x}} = -\frac{1}{a^2}\nabla\Psi\,, \tag{4.23}$$

we find that

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= -\frac{1}{a^2}\int dt\,\frac{D}{a}\nabla\psi(\mathbf{x}_0) = \frac{\dot{D}(t)}{4\pi G\rho_{b,0}}\,\nabla\psi(\mathbf{x}_0) && (4.24)\\
\mathbf{x}(t) &= \mathbf{x}_0 - \int\frac{dt}{a^2}\int dt\,\frac{D}{a}\,\nabla\psi(\mathbf{x}_0) = \mathbf{x}_0 - \frac{D}{4\pi G\rho_{b,0}}\,\nabla\psi(\mathbf{x}_0)\,. && (4.25)
\end{aligned}$$

To see this, firstly check that Eq. (4.24) is indeed the solution to Eq. (4.23). Now, taking the first and second derivative of the double integral $F(t) = \int\frac{dt}{a^2}\int dt\,\frac{D}{a}$, we find that

$$\ddot{F} + 2H\,\dot{F} = \frac{D}{a^3}\,. \tag{4.26}$$

Since we know that $\delta(\mathbf{x},t) = D(t)\,\delta(\mathbf{x})$ is the solution to Eq. (4.19), $\ddot{D} + dH\,\dot{D} = 4\pi G\,\rho_{b,0}D$, hence $F = D/(4\pi G\rho_{b,0})$.

How accurate is the Zel'dovich approximation? Equation (4.25) shows that we approximate the orbit of each particle as a straight line, traversed with velocity $a\dot{\mathbf{x}} \propto a\dot{D}$. We can use mass conservation to compute the density as a function of time:

$$\rho\,d^3\mathbf{x} = (1+\delta)\,\rho_0\,d^3\mathbf{x} = \rho_0 d^3\mathbf{x}_0\,, \tag{4.27}$$

hence $1+\delta$ is the Jacobian of the transformation $\mathbf{x}_0 \to \mathbf{x}$,

$$1 + \delta = \frac{1}{det(\delta_{ij} - (D/4\pi G\rho_{b,0})(\partial^2\psi(0)/\partial x_i/\partial x_j)))} \,. \qquad (4.28)$$

The matrix $(1/4\pi G\rho_{b,0})(\partial^2\psi(0)/\partial x_i/\partial x_j)$ is symmetric, hence we can compute its eigenvalues λ_i, $i = 1, 3$, to find

$$1 + \delta = \frac{1}{(1 - \lambda_1 D)\,(1 - \lambda_2 D)\,(1 - \lambda_3 D)} \,, \qquad (4.29)$$

where we can choose $\lambda_1 \geq \lambda_2 \geq \lambda 3$. For time such that $\lambda_1 D = 1$, the density becomes infinite. Zel'dovich noted that this corresponds to the collapse of a *sheet*, and it shows that any collapsing structure will first collapse as a sheet, before collapsing along the second, and finally third axis.

The great strength of the Zel'dovich approximation is that the description is in fact exact in one dimension. And since the 3D object will first collapse in 1D anyway, it is a much more accurate description of the non-linear evolution of the density field than you would at first expect. Note that δ need not be small in the Zel'dovich approximation: what needs to be small is the tidal force (which makes the particle's trajectory deviate from a straight line) and the *acceleration* along its trajectory.

4.4 Angular momentum

A beautiful application of the Zel'dovich approximation is the generation of angular momentum of galaxies. Recall that in spiral galaxies, it is the angular momentum of the stars and the gas that opposes collapse.

The angular momentum $\mathbf{L}$ of the matter contained at time t in a volume a^3V in co-moving space $\mathbf{x} = \mathbf{r}/a$ is

$$\mathbf{L}(t) = \int_{a^3V} d\mathbf{r}\,\rho\,(\mathbf{r} \times \mathbf{v})\,, \qquad (4.30)$$

where the integral is assumed to be performed with respect to the centre of mass of the galaxy, so $\int_{a^3V} d\mathbf{r}\,\rho\,\mathbf{r} = 0$.

As usual $\rho(\mathbf{r}) = \rho_b(1 + \delta(\mathbf{r}))$ is the density at position $\mathbf{r}$, and $\mathbf{v} = a\dot{\mathbf{x}}$ is the velocity. Note that the Hubble component $\dot{a}\mathbf{x}$ of the velocity does not contribute to $\mathbf{L}$, since $\mathbf{x} \times a\mathbf{x} = 0$.

The previous integral over $\mathbf{x}$ (in 'Eulerian' space) can also be written as an integral in Lagrangian space $\mathbf{x}_0$, when we introduce the displacement field $\mathbf{S}(\mathbf{x}_0, t)$,

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{q} + \mathbf{S}(\mathbf{q}, t) \\ &= \mathbf{q} + D(t)\, \mathbf{S}_Z(\mathbf{q})\,, \end{aligned} \tag{4.31}$$

where we have made the Zel'dovich approximation in the second line (hence the subscript Z), put $\mathbf{x}(t = t_0) \equiv \mathbf{q}$, and

$$\mathbf{S}_Z(\mathbf{q}) = -\frac{1}{4\pi G \rho_{b,0}}\, \nabla\psi(\mathbf{q})\,. \tag{4.32}$$

Expressing mass conservation, $(1 + \delta)\, d\mathbf{x} = d\mathbf{q}$, we obtain

$$\begin{aligned} \mathbf{L}(t) &= a^4\, \rho_b \int_{V_0} d\mathbf{q} \left[(\mathbf{q} + D\mathbf{S}_Z) \times a\dot{D}\mathbf{S}_Z\right] \\ &= a^5 \dot{D}\, \rho_b \int_{V_0} d\mathbf{q}\ \mathbf{q} \times \mathbf{S}_Z\,, \end{aligned} \tag{4.33}$$

since $\mathbf{S}_Z \times \mathbf{S}_Z = 0$. We can make progress by expanding the displacement field $\mathbf{S}_Z(\mathbf{q})$ in Taylor series from the centre of mass $\mathbf{q}_0$ of the object, hence for the i-th component

$$\begin{aligned} \mathbf{S}_{Z,i}(\mathbf{q}) &= -\frac{1}{4\pi G \rho_{b,0}} \frac{\partial \psi}{\partial q_i}(\mathbf{q}) \\ &= -\frac{1}{4\pi G \rho_{b,0}} \left[\frac{\partial \psi}{\partial q_i}(\mathbf{q}_0) + \mathbf{q}_j\, \frac{\partial^2 \psi}{\partial q_i \partial q_j}(\mathbf{q}_0) + \cdots\right]. \end{aligned} \tag{4.34}$$

The last equation defines the centre of mass force $\mathcal{F}$, and the deformation tensor $\mathcal{T}$, as

$$\begin{aligned} \mathcal{F}_i &\equiv -\frac{1}{4\pi G \rho_{b,0}} \frac{\partial \psi_0}{\partial q_i}(\mathbf{q}_0) \\ \mathcal{T}_{i,j} &\equiv -\frac{1}{4\pi G \rho_{b,0}} \frac{\partial^2 \psi_0}{\partial q_i \partial q_j}(\mathbf{q}_0)\,, \end{aligned} \tag{4.35}$$

hence

$$\mathbf{S}_{Z,i}(\mathbf{q}) = \mathcal{F}_i + \mathbf{q}_j\,\mathcal{T}_{i,j} + \cdots\,. \tag{4.36}$$

Both $\mathcal{F}$ and $\mathcal{T}$ are computed at the position $\mathbf{q}_0$ of the centre of mass, and so are constants in the integral $\int_{V_0} d\mathbf{q}$.

Substituting the Taylor expansion yields

$$\begin{aligned}
\mathbf{L}_\alpha(t) &= a^5\dot{D}\rho_b\,\epsilon_{\alpha\beta\gamma}\int_{V_0} d\mathbf{q}\,\mathbf{q}_\beta\,\left(\mathcal{F}_\gamma + \mathbf{q}_\delta\,\mathcal{T}_{\delta\gamma}\right) \\
&= a^5\dot{D}\rho_b\,\epsilon_{\alpha\beta\gamma}\int_{V_0} d\mathbf{q}\;\mathbf{q}_\beta\,\mathbf{q}_\delta\,\mathcal{T}_{\delta\gamma} \\
&= a^2\dot{D}\;\;\epsilon_{\alpha\beta\gamma}\,\mathcal{I}_{\beta\gamma}\,\mathcal{T}_{\delta\gamma}\,.
\end{aligned} \tag{4.37}$$

Here,

$$\mathcal{I}_{\beta\gamma} \equiv (a^3\rho_b)\int_{V_0} d\mathbf{q}\;\; q_\beta\,q_\gamma\,, \tag{4.38}$$

is the *inertia* tensor of the Lagrangian volume V_0, and ϵ is the usual anti-symmetric pseudo tensor, with $\epsilon_{123} = 1$. Note that $\rho_b\,a^3$ is time-independent. The term in $\mathcal{F}$ does not contribute because $\int d\mathbf{q}\;q_\alpha = 0$, since we use a system of axes where the centre of mass is in the origin.

The final result is thus

$$\mathbf{L}_\alpha(t) = a^2\dot{D}\;\;\epsilon_{\alpha\beta\gamma}\,\mathcal{I}_{\beta\delta}\,\mathcal{T}_{\gamma\delta}\,. \tag{4.39}$$

From this we see that angular momentum grows when the principle axes of the inertia tensor $\mathcal{I}$ are not aligned with those of the deformation tensor $\mathcal{T}$.

In an Einstein-de Sitter Universe, $a = D \propto t^{2/3}$, hence $\mathbf{L}(t) \propto a^2\dot{D} \propto t$: the angular momentum grows linearly in time.

We can estimate the *final* angular momentum of a forming galaxy, as its linear angular momentum $\mathbf{L}$, evaluated at the time t_m when the over density $\delta = -D\nabla^2\psi \sim 1$. Here I've used Poisson's equation $\nabla^2\phi \propto \delta$. Since the inertia tensor of an object of mass M and radius R is $\mathcal{I} \propto MR^2 \propto M^{5/3}$, this gives

$$\mathbf{L} \approx a^2(t_m)\,\frac{\dot{D}(t_m)}{D(t_m)}\,(D(t_m)\nabla^2\psi)\,M^{5/3} \tag{4.40}$$

which is $\propto t_m^{1/3} M^{5/3}$ for an EdS Universe. Note that this implies that $L/M \propto M^{2/3}$: the angular momentum per unit mass increases with mass. Numerical simulations show that this is a reasonable description of what actually happens during the non-linear formation of objects.

4.5 Numerical simulations

4.5.1 Introduction

Computer simulations have revolutionised the study of structure formation in the Universe, almost to the extent that we 'know' how structure grows in a CDM Universe, at least in the absence of gas. Such numerical calculations follow the linear and later non-linear collapse of structures, starting from Gaussian initial conditions with a specified transfer function, and cosmological parameters. They use the Zel'dovich approximation to generate particle positions at a time when the density field is everywhere linear, and integrate the equations of motion to follow the collapse into the non-linear regime.

The general appearance of the density field is a filamentary pattern of over dense structures, delineating almost spherical low density 'voids'. Denser, nearly spherical structures appear along and at the intersection of these filaments: these are dark matter haloes of galaxies, and the haloes of more massive groups and clusters.

The filamentary pattern comes from the collapse of 1D structures, the Zel'dovich sheets, along a second axis. It is not easy to see the actual sheets in simulations, although they are there. An example is shown in figure 4.2.

4.5.2 Co-moving variables

Although it is possible to implement a numerical simulation code in terms of the 'physical coordinates' $\mathbf{r}, \mathbf{v}$, it may be more efficient to implement them in terms of a form of the co-moving coordinates $(\mathbf{x}, \dot{\mathbf{x}})$, and in a similar vein, in terms of some (to be defined) co-moving density, pressure and temperature. For example suppose we wrote our equations in terms of the following co-moving density $\hat{\rho} \equiv \rho\, a^3$. In the limiting case in which the Universe stayed uniform, $\rho \propto a^{-3}$, and $\hat{\rho}$ remained constant. So this choice of co-moving

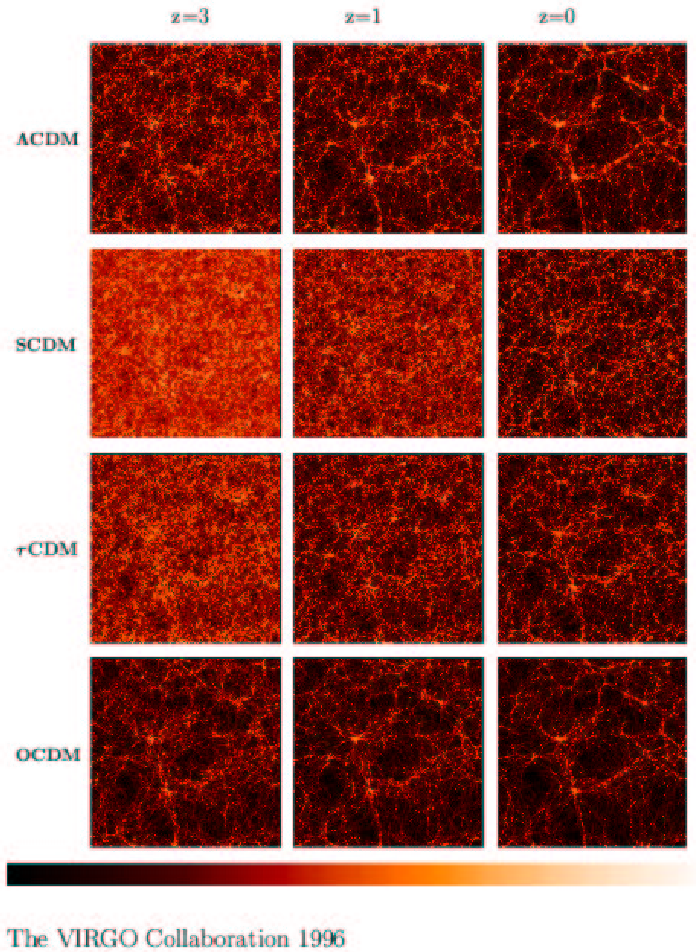

Figure 4.2: Particle plots of the dark matter density in cold dark matter dominated models. The simulation box is 240Mpc/h on a side, the thickness of the shown slices is around 8Mpc/h. The particle distribution is very filamentary, a consequence of the large amount of large scale power in this model. The higher density filaments delineate nearly spherical under dense voids. Very high density haloes occur on the intersection between filaments. Different redshifts ($z = 3$, 1, 0) are shown from left to right. Plots from top to bottom are (1) a Universe with cosmological constant ($\Omega_m = 0.3, \Omega_\Lambda = 0.3$), (2) Einstein-de Sitter Universe ($\Omega_m = 1.0, \Omega_\Lambda = 0.0$), (3) a cosmology with a tilted power-spectrum ($\Omega_m = 1.0, \Omega_\Lambda = 0.0$), (4) an $\Omega_m = 0.3$ Open model ($\Omega_m = 0.3, \Omega_\Lambda = 0.0$). (Jenkins et al, 1998 Astrophysical Journal,499,20-40)

variable would mean that the dynamic range over which $\hat{\rho}$ varies is much less (and hence the numerical calculation less affected by round-off errors). We will start our discussion for the following version of Eqs.(3.21)

$$\begin{aligned}
\dot{\rho}_0\,(1+\delta) + \nabla\left[\rho_0(1+\delta)\dot{\mathbf{x}}\right] &= 0 \\
\ddot{\mathbf{x}} + 2H\dot{\mathbf{x}} + (\dot{\mathbf{x}}\cdot\nabla)\dot{\mathbf{x}} &= -\frac{1}{a^2}\nabla\Psi - \frac{1}{a^2}\,\frac{\nabla p}{\rho} \\
\dot{u} + 3H\frac{p}{\rho} + (\dot{\mathbf{x}}\cdot\nabla)u &= -\frac{p}{\rho}\nabla\dot{\mathbf{x}} \\
\nabla^2\Psi &= 4\pi G\rho_0\delta a^{-1}\,. \qquad (4.41)
\end{aligned}$$

which came from writing the density as $\rho(\mathbf{x},t) = \rho_0\,a(t)^{-3}\;(1+\delta(\mathbf{x},t))$. We need to integrate these equations together with the Friedmann equations (2.12) that determine the evolution of the scale factor $a(t)$.

To make progress, we will need to pick a set of variables in which to write the equations (for example $(\mathbf{x},\dot{\mathbf{x}})$, but also a set of units (dimensions).

4.5.3 The Gadget-II simulation code

This section goes into more detail into how the cosmological equations of motion discussed above can be implemented in a numerical code. In particular we will look in more detail at how it is done in Volker Springel's Gadget-II.