

Stellar Dynamics

Stellar dynamics describes systems of many point mass particles whose mutual gravitational interactions determine their orbits. These particles are usually taken to represent stars in small GALAXY CLUSTERS with about 10^2 – 10^3 members, or in larger GLOBULAR CLUSTERS with 10^4 – 10^6 members or in GALACTIC NUCLEI with up to about 10^9 members or in galaxies containing as many as 10^{12} stars. Under certain conditions, stellar dynamics can also describe the motions of galaxies in clusters, and even the general clustering of galaxies throughout the universe itself. This last case is known as the cosmological many-body problem.

The essential physical feature of all these examples is that each particle (whether it represents a star or an entire galaxy) contributes importantly to the overall gravitational field. In this way, the subject differs from CELESTIAL MECHANICS where the gravitational force of a massive planet or star dominates its satellite orbits. Stellar dynamical orbits are generally much more irregular and chaotic than those of celestial mechanical systems.

Consequently, the description of stellar dynamical systems is usually concerned with the statistical properties of many orbits rather than with the detailed positions and velocities of an individual orbit. Thus it is not surprising that the KINETIC THEORY OF GASES developed by MAXWELL, Boltzmann and others in the late 19th century was adapted by astrophysicists such as Jeans to stellar dynamics in the early 20th century. Subsequent results in stellar dynamics contributed to the first analyses of kinetic plasma physics in the 1950s. Then rapid evolution of plasma theory in the second half of the 20th century, stimulated partly by prospects of controlled thermonuclear fusion, in turn contributed to stellar dynamics. This was an especially productive interdisciplinary interaction.

After describing basic physical processes such as timescales, relaxation processes, dynamical friction and damping, this article derives the virial theorem and mentions some applications, discusses distribution functions and their evolution through the collisionless Boltzmann equation and the BBGKY hierarchy, and outlines the thermodynamic descriptions of finite and infinite gravitating systems. The emphasis here is on fundamental physics rather than on detailed models.

Basic ideas

We start with simple ideas that are common to most stellar dynamical systems. For point masses to represent their components, physical collisions must be rare. In a system of objects, each with radius d , this means that the total internal volume of all the objects must be much less than the volume over which they swarm. Two spherical objects of radius d will have an effective radius $2d$ for a grazing collision whose cross section is therefore $\sigma = 4\pi d^2$. If there is a number density n of these objects and they move on

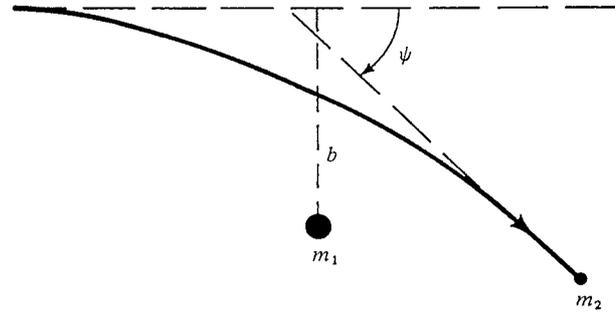


Figure 1. The deflection of a star m_2 by a more massive star m_1 , schematically illustrating two-body relaxation.

random orbits, their mean free path to geometric collisions is

$$\lambda_G \approx 1/n\sigma \approx \frac{R^3}{3Nd^3}d \quad (1)$$

where R is the radius of a spherical system containing N objects distributed approximately uniformly.

This has two easy physical interpretations. First, the average number of times an object can move through its own diameter before colliding is essentially the ratio of the cluster's volume to that occupied by all the stars. Second, the number of cluster radii the object can traverse before colliding is essentially the ratio of the projected area of the cluster to that of all the objects. In many astronomical systems, these ratios are very large. As examples, 10^5 stars in a globular cluster of 10 pc radius have $\lambda_G/R \approx 3 \times 10^{11}$ and 10^3 galaxies in a cluster of 3 Mpc radius have $\lambda_G/R \approx 30$. Therefore stellar dynamics is a good approximation over a wide range of conditions. It may, however, break down in the cores of realistic systems where only a few objects dominate at the very center and orbits are more regular.

Although geometric collisions may be infrequent, gravitational encounters are common. These occur when one object passes by another, perturbing both orbits. Naturally, in a finite system all the objects are passing by each other all the time, so this process is continuous. In a system which is already fairly stable, most perturbations are small. However, their cumulative effects over long times can be large and affect the evolution of the system significantly. To see how this works, we introduce the fundamental notion of a 'stellar dynamical relaxation time'. This is essentially the timescale for a dynamical quantity such as a particle's velocity to change by an amount approximately equal to its original value.

As an illustration of the general principle, consider two-body relaxation. Suppose, as in figure 1, that a massive star m_1 , deflects a much less massive star m_2 . Initially, m_2 moves with velocity v perpendicular to the distance b (the impact parameter) at which the undeflected orbit would be closest to m_1 .

There is a gravitational acceleration Gm_1b^{-2} which acts for an effective time $2bv^{-1}$ and produces a component

of velocity

$$\Delta v \approx \frac{2Gm_1}{bv} \quad (2)$$

approximately perpendicular to the initial velocity. Since $\Delta v \ll v$, the effects are linear and they give a scattering angle $\psi \approx \Delta v/v$. A more exact version follows the orbits in detail, but this is its physical essence. It shows that large individual velocity changes, $\Delta v \approx v$, and large scattering angles occur when two objects are so close that the gravitational potential energy, Gm_1m_2/b , approximately equals the kinetic energy, $m_2v^2/2$. Such close encounters are rare. Typical encounters in a spherical system of total mass M and radius R containing N objects have an impact parameter about equal to the mean separation, $b \approx RN^{-1/3}$. The average mass $m_1 \approx M/N$, and the initial velocity is given by the approximate balance of the system's total kinetic and potential energy, $v^2 \approx GM/R$. (This last relation follows from the virial theorem below.) Therefore from equation (2)

$$\frac{\Delta v}{v} \approx \psi \approx N^{-2/3} \quad (3)$$

for $N \gg 1$, and most gravitational encounters involve little energy or momentum exchange. Exceptions may occur in the centers of clusters, particularly among more massive particles, where the core is relatively isolated and the effective value of N is low.

Although most individual deflections are small, their cumulative effects are not. Over long times, uncorrelated deflections from the nearly randomly moving orbits cause each particle's velocity to randomly walk around its original value. Thus the root-mean-square velocity increases from its initial value in proportion to the square root of the number of encounters, or the square root of the time. To estimate this change, we multiply the square of each velocity change from equation (2) by the number of encounters with a given velocity and impact parameter during a unit time and then integrate over the whole range of velocities, impact parameters and times of interest. Finally setting $(\Delta v)^2 = v^2$ gives the stellar dynamical relaxation time for the cumulative small changes to modify the velocity by an amount equal to its initial value. For homogeneous systems near equilibrium with $GM \approx Rv^2$ and a Maxwellian distribution of velocities, this is

$$\tau_R \approx \frac{0.2N}{\ln(N/2)} \tau_c \quad (4)$$

where $\tau_c = R/v \approx (G\bar{\rho})^{-1/2}$ is the dynamical crossing time for a particle with average velocity and $\bar{\rho}$ is the average density.

In globular clusters, galaxies and rich clusters of galaxies $N \gtrsim 10^3$ so $\tau_R \gg \tau_c$. Cumulative small deflections do not amount to much over any one orbital period, but their secular effect can dominate after many orbits. The relaxation time τ_R will vary depending on the geometry, range of masses and velocity and density distributions

within a system. For most systems, τ_R exceeds the lifetime of the system, so it cannot have been the primary process for a system's formation and currently relaxed regular roughly spherical appearance.

Systems in which $\tau_R \gg \tau_c$ are essentially 'collisionless' in the sense that their nearly smooth global gravitational force dominates the average orbits of their particles. Other systems in which strong local fluctuations in gravitational forces dominate particle orbits are 'collisional', since these fluctuations produce large-angle scatterings of the orbits. (This standard terminology does not refer to bodily collisions of objects.) Realistic astronomical systems usually combine aspects of both of these idealized categories. For example, local gas clouds, star clumps or dark matter inhomogeneities may scatter stellar orbits from their average smooth paths in GALAXIES.

Even for maintaining a relaxed system near a state of equilibrium, gravitational encounters cannot be the whole story. If they were, the constant average increase in the root-mean-square velocities of all the particles would eventually cause them all to escape, defying the conservation of energy. A second, balancing, process must operate. This is dynamical friction.

Imagine yourself on a particle moving faster than the average speed of the particles around it. In a stationary reference frame attached to your particle, the other particles will appear to stream by. More particles will stream by in the direction you are moving toward than in other directions. The orbits of these excess particles will be deflected slightly by the gravity of your particle. This deflection causes a slight convergence of the orbits behind your particle, opposite the direction of its motion. On average, therefore, there is a statistical excess of particles behind yours. Their excess gravity decelerates the motion of your particle, and tends to reduce it to the average velocity. Conversely, if the speed of your particle is slower than average, there will be a statistical excess of particles passing it in your direction of motion. Their orbits will converge slightly in front of the motion of your particle, and their excess gravitational force will speed up your particle. Thus dynamical friction is a great leveller of velocities. The balance between dynamical friction and two-body acceleration keeps the system close to equilibrium. Therefore τ_R must also be the approximate timescale for relaxation by dynamical friction. Mathematically, a second-order non-linear partial differential-integral equation, known as the Fokker-Planck equation, describes this balance, and it is usually solved numerically.

Dynamical friction also plays another important role in stellar dynamics. If a massive object M (such as a high-mass star in a galaxy, or a large black hole in a galactic nucleus or even a small galaxy in the halo of another larger galaxy) moves rapidly through a field of much less massive objects, m , dynamical friction will cause it to lose kinetic energy, fall deeper into the surrounding gravitational well, speed up while falling, lose more energy and consequently spiral into the center of the larger

system. This helps promote the merging of galaxies, for example (see GALAXIES: INTERACTIONS AND MERGERS). The relaxation time τ_R of equation (4) is decreased significantly by a factor of order m/M . This is because the large mass difference induces *gravitational excitations*, known as *grexons* (also from the Latin ‘grex’ meaning ‘herd’ or ‘flock’): a collective gravitational excitation in the wake of a massive particle moving through a system of less massive particles. It generally takes the form of a region of statistical overdensity among the less massive particles passing through the wake. These gravitating collective modes of many small masses interact with each other, enhancing their lifetimes and producing a larger effective statistical condensation behind M through which particles of mass m move but linger longer than they would for $M \approx m$. These grexons arise when $M \gg m$ and amplify the gravitational drag.

Other forms of collective excitations can also amplify relaxation in systems which are initially far from equilibrium. This collective relaxation—of which ‘violent relaxation’ is an extreme case—is much faster than τ_R and closer to the crossing timescale τ_c . Let us start by thinking about a relatively simple case. Suppose one star of mass m initially has zero velocity at the edge of an already relaxed cluster of stars with total mass M and radius R . Details of its orbit would depend on the density distribution of the cluster and the building up of small perturbations, but the star would roughly follow the equation of motion $m\ddot{r} \approx GmMR^{-2}$. Since $\ddot{r} \approx R/t^2$, this would give a characteristic infall timescale $\sim (G\bar{\rho})^{-1/2}$, which is just the crossing time τ_c . We may think of this as the star being scattered by the average force of the entire cluster, i.e. by the ‘mean field’. As the star falls inward, the small discrete perturbations of nearby stars will also scatter it, and this ‘fluctuating field’ will impart some net angular momentum to the star so that it is unlikely to reach the exact center of the cluster. Instead, it will join the other stars in their complicated orbits. The net relaxed orbit of the star comes from the joint effects of the smooth global mean field and the two-body relaxation of the local fluctuating field.

Now let us make the process a little more exciting. Suppose the cluster is not itself relaxed. Then it will not be quasi-static. Parts of it will be falling together, interpenetrating, separating, streaming in different directions and generally flopping around. These represent larger-scale fluctuations—collective modes—which will scatter the incoming star in addition to its scattering by the global mean field and the shorter-scale fluctuations of nearby individual stars. Moreover, the large-scale fluctuations will also be scattering and interacting with each other. With high-amplitude large-mass collective modes fluctuating over a wide range of lengthscales, and on corresponding timescales shorter than τ_c for the entire cluster, individual orbits will relax on timescales closer to τ_c than τ_R . This is collective relaxation, and it may apply to the formation of clusters of stars, galaxies and clusters of galaxies, particularly if they build up by the merging of smaller systems.

Violent relaxation occurs in the extreme limit when collective modes are so chaotic on all scales that each of them lasts for only a short time but is quickly replaced by another such mode elsewhere in the system. None of these modes correlates with one another. Each star moves in a mean field which changes quickly in time as well as in space. Consequently, the energy of each individual star along its orbit is not conserved; only the energy of the entire system remains constant. If this process could continue indefinitely, the velocity distribution of the stars would generally become similar to the Maxwell–Boltzmann distribution for a perfect gas after a timescale τ_c .

However, it does not. On this same gravitational free-fall timescale, τ_c , damping mechanisms destroy the ideal conditions for violent relaxation. In stellar dynamics, as in other systems, damping mechanisms reduce departures from an equilibrium state. Most damping is characterized by dynamical dissipation—the roughly random transfer of relatively ordered (low-entropy) energy into relatively disordered (high-entropy) energy. The transfer may occur among ‘particles’, as when a stellar cluster or a forming cluster of galaxies ejects a high-energy member so that the remaining cluster becomes more tightly bound. Another example of particle dissipation is the phase mixing of orbits. Generally this involves interactions of stars at slightly different phases of the same, otherwise unperturbed, orbit. As a simple illustration, suppose an isolated spherical cluster of stars, initially completely at rest, starts falling together. The stars will not simply all plunge together into the center. Instead, stars at different distances from the center, i.e. at different phases of their radial orbits, will perturb each other. These perturbations can be large among stars with low relative velocities, and can occur on the dynamical crossing timescale τ_c . Most stars will acquire some angular momentum from these perturbations, although the cluster’s net angular momentum remains zero. This further mixes the phase of the orbits, dissipating some of the initially highly ordered radial infall velocities into more random transverse velocities. Eventually, a component of random kinetic energy builds up as a form of heat whose effective pressure resists further collapse of the cluster on the free-fall timescale τ_c . Similar phase mixing of orbits, combined with heat generated by collective relaxation itself, can occur to damp more general collective motions such as those of violent relaxation. More gentle phase mixing can also occur in relaxed galaxies over longer timescales to mix up streaming motions of stars.

Dissipation and damping may also result when particles and waves interact. The waves are moving periodic density perturbations. The best-known case was first found for plasmas by Landau and then applied to stellar dynamics. It occurs even in idealized collisionless systems where particles do not scatter one another significantly. The physical reason for collisionless damping arises from the detailed interaction of a wave with the orbits of background stars which are not part of

the wave. Thus this process would not show up in a pure continuum approximation. What happens is that some of the background stars will move a bit faster than the wave, others a bit slower, even though their average background number density is constant. The wave exerts a force on these stars and thus exchanges energy with them. On average, the wave gains energy from the fast-moving stars, which therefore amplify the wave, and loses energy to the slow-moving stars, which damp the wave. The net result depends on whether there are more fast or slow stars with velocities near the phase velocity $v = \omega/k$ of the wave. Moreover, the strength of the growth or damping depends on the total net number of fast or slow stars involved. A detailed analysis of this result is complex but, for a roughly Maxwellian velocity distribution of background stars, damping is greatest for waves whose phase velocity ω/k is near the velocity dispersion $\langle v \rangle$ of the background stars. These waves decay on a timescale $\lambda/\langle v \rangle$ during which the stars are able to move through the wave.

Phase mixing, Landau damping and other processes such as trapping by clusters, tidal disruption and small-angle scattering all combine with violent relaxation into a form of collective relaxation which randomizes velocities on a timescale $\sim (G\bar{\rho})^{-1/2}$, provided that the system starts out very far from its eventual quasi-equilibrium state. What is left is a system in quasi-equilibrium with a roughly Maxwellian velocity distribution. No direct collisions are responsible for this end result—unlike the case of a perfect gas—only the non-linear encounters among particles and collective modes.

The virial theorem

This is perhaps the most astronomically important dynamical property of a quasi-equilibrium system. Therefore we describe the virial theorem in some detail. With it, we can estimate the system's total mass from observations of the positions and velocities of its particles. The virial theorem is essentially just a position moment of the self-consistent gravitational equations of motion within the system. To illustrate it in the simplest case, consider a satellite of mass $m^{(\alpha)}$ in a circular orbit around a much more massive object of mass $m^{(\beta)}$. The balance between centrifugal and gravitational forces gives $m^{(\alpha)}v^2/r = Gm^{(\alpha)}m^{(\beta)}/r^2$. Multiplying through by r gives $2K + W = 0$ where $K = m^{(\alpha)}v^2/2$ is the kinetic energy and $W = Gm^{(\alpha)}m^{(\beta)}/r^2$ is the gravitational potential energy. In this circular orbit the instantaneous value and the time average of the energy are identical. For more complicated configurations, we might suspect that this result will still hold for the more general time averages of K and W . It does.

The most general Newtonian equations of motion for a system of gravitating particles are

$$\begin{aligned} \frac{d}{dt}(m^{(\alpha)}v_i^{(\alpha)}) &= F_i^{(\alpha)} = m^{(\alpha)} \frac{\partial \varphi}{\partial x_i^{(\alpha)}} \\ &= -Gm^{(\alpha)} \sum_{\beta \neq \alpha} m^{(\beta)} \frac{x_i^{(\alpha)} - x_i^{(\beta)}}{|\mathbf{x}^{(\alpha)} - \mathbf{x}^{(\beta)}|^3} \end{aligned} \quad (5)$$

for the i th components of position and velocity, where the gravitational potential

$$\varphi = G \sum_{\beta \neq \alpha} \frac{m^{(\beta)}}{|\mathbf{x}^{(\alpha)} - \mathbf{x}^{(\beta)}|} \quad (6)$$

results from all the other particles. This derivation is general enough to let the particle masses vary with time, by mass loss or accretion, isotropically as we will assume, or even anisotropically with a net 'rocket effect'.

Now multiply equation (5) by $x_j^{(\alpha)}$ and sum over α to take its lowest-order position moment. Since α and β are just dummy indices, we could just as validly multiply through by $x_j^{(\beta)}$. Equivalently, and more elegantly, we could represent the right-hand side as half the sum of both these multiplications, i.e. as

$$W_{ij} = -\frac{G}{2} \sum_{\alpha} \sum_{\beta \neq \alpha} m^{(\alpha)} m^{(\beta)} \frac{(x_i^{(\alpha)} - x_i^{(\beta)})(x_j^{(\alpha)} - x_j^{(\beta)})}{|\mathbf{x}^{(\alpha)} - \mathbf{x}^{(\beta)}|^3} \quad (7)$$

which is symmetric to an interchange of the α and β particles. This is known as the potential energy tensor, W_{ij} .

Rewriting the left-hand side of equation (5) multiplied by $x_j^{(\alpha)}$ gives

$$\begin{aligned} \sum_{\alpha} x_j^{(\alpha)} \frac{d}{dt}(m^{(\alpha)}v_i^{(\alpha)}) &= \frac{d}{dt} \sum_{\alpha} m^{(\alpha)} x_j^{(\alpha)} v_i^{(\alpha)} - \sum_{\alpha} m^{(\alpha)} v_i^{(\alpha)} v_j^{(\alpha)} \\ &= \frac{d}{dt} \sum_{\alpha} \frac{1}{2} m^{(\alpha)} [(x_j^{(\alpha)} v_i^{(\alpha)} + x_i^{(\alpha)} v_j^{(\alpha)}) \\ &\quad + (x_j^{(\alpha)} v_i^{(\alpha)} - x_i^{(\alpha)} v_j^{(\alpha)})] - \sum_{\alpha} m^{(\alpha)} v_i^{(\alpha)} v_j^{(\alpha)}. \end{aligned} \quad (8)$$

The terms in parentheses separated by a minus sign are antisymmetric, and since all the other contributions to the moment of equation (5) are symmetric, these antisymmetric terms must be zero. This proves that angular momentum is conserved, since these antisymmetric terms are just the total angular momentum of the isolated system. The symmetric terms in equation (8) may be written as

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{\alpha} m^{(\alpha)} (x_j^{(\alpha)} v_i^{(\alpha)} + x_i^{(\alpha)} v_j^{(\alpha)}) &- \sum_{\alpha} m^{(\alpha)} v_i^{(\alpha)} v_j^{(\alpha)} \\ &= \frac{1}{2} \frac{d^2}{dt^2} \sum_{\alpha} m^{(\alpha)} x_i^{(\alpha)} x_j^{(\alpha)} \\ &- \frac{1}{2} \frac{d}{dt} \sum_{\alpha} \dot{m}^{(\alpha)} x_i^{(\alpha)} x_j^{(\alpha)} - \sum_{\alpha} m^{(\alpha)} v_i^{(\alpha)} v_j^{(\alpha)}. \end{aligned} \quad (9)$$

Now each of the three summations over α on the right-hand side of equation (9) has a physical interpretation. The first summation is the usual inertia tensor I_{ij} , the second we shall call the mass variation tensor J_{ij} and the third is twice the kinetic energy tensor T_{ij} which incorporates all the motions of all the particles. Combining equations (7) and (9) for the moment of equation (5) then gives the tensor

virial theorem for a system of gravitationally interacting particles:

$$\frac{1}{2} \frac{d^2 I_{ij}}{dt^2} - \frac{1}{2} \frac{d}{dt} J_{ij} = 2T_{ij} + W_{ij}. \quad (10)$$

This is an exact result, without any approximations so far. It provides an overall constraint on the system's evolution. Further constraints follow from taking higher-order moments of the equations of motion. Multiplying equation (5) by $x^m v^n$ and summing gives the $(m+n)$ th combined spatial and velocity moment. This forms an infinite sequence of virial moments. The more moments are used, the tighter are the constraints on the system's evolution. All the moments taken together usually become equivalent to a complete solution of the original equations of motion (5).

In practice, most astronomical calculations are content with a simplified version of the lowest-order virial equation (5). If the mass loss rate $\dot{m}^{(\alpha)} = 0$, then $J_{ij} = 0$ and the form reduces to that quoted in many texts. For the special case of mass loss proportional to the mass, $\dot{m}^{(\alpha)} = f(t)m^{(\alpha)}(t)$, the mass loss tensor is proportional to the moment of inertia tensor. A great simplification follows by taking the time average of the virial theorem. For example, the time average of the first term gives

$$\left\langle \frac{d^2 I}{dt^2} \right\rangle = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \frac{dI}{dt} dt = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} [I(\tau) - I(0)]. \quad (11)$$

This time average can be zero either if the system is localized in position and velocity space so that $\dot{I}(\tau)$ has an upper bound for all τ or if the orbits are periodic so that $\dot{I}(\tau) = \dot{I}(0)$. Similarly the time average of J_{ij} can be zero if $\dot{m}(t)$ does not increase as fast as t for $t \rightarrow \infty$. The terms on the right-hand side of equation (10) remain non-zero over any time average, so

$$2\langle T_{ij} \rangle + \langle W_{ij} \rangle = 0. \quad (12)$$

The usual (contracted) form of the virial theorem is obtained by setting $i = j$ and summing over $i = 1, 2, 3$ to give

$$2\langle T \rangle + \langle W \rangle = 0 \quad (13)$$

where T and W are the entire kinetic and potential energies of the system.

The virial theorem describes reasonably bound, quasi-stable clusters of stars and galaxies. Its usual application is to estimate the mass of these clusters by writing equation (13) in the approximate form

$$V^2 = \gamma \frac{GM}{R}. \quad (14)$$

This is similar to the form for a circular satellite orbit discussed earlier, but now V^2 is the velocity dispersion of objects in the cluster, R is the cluster's radius and M is its total mass. The constant γ is usually of order unity and depends on the precise operational definitions of V , R and M , especially if the particle

masses are not identical. Other factors which influence γ when trying to deduce M are projection effects in transverse positions, loss of one position and two velocity components, departures of time averages from the instantaneous snapshots of observations, questions of cluster membership and selection of a limited number of particles for observation. Each of these effects depends on the particular circumstances of an individual cluster. Their combination introduces considerable uncertainty into γ , but this can often be estimated with computer simulations.

Virial estimates of the masses of CLUSTERS OF GALAXIES have been made since the 1930s. A succession of more accurate results with more complete samples and better simulations, which continues to the present, revealed clear discrepancies between the virial mass of many clusters and their mass estimated from the luminosities of their individual galaxies. Unless cluster formation proceeded by a radically different route from the condensation processes discussed currently, this discrepancy suggests that the amount of 'DARK MATTER' in clusters is about five times the amount of luminous matter in galaxies. This gives roughly one-fourth of the total mass needed for a closed Einstein–Friedmann universe. The form of this dark matter and its total amount are two of the main problems of modern astronomy.

Distribution functions

Rather than dealing with the positions and motions of each identifiable particle individually, stellar dynamics often finds a less detailed description based on distribution functions to be more useful and solvable. These distributions represent the probability that any arbitrary particle in the system has a particular property of interest. Usually stellar dynamics is concerned with the single-particle distribution $f(\mathbf{r}, \mathbf{v})$ for the number of particles having position \mathbf{r} and velocity \mathbf{v} . Integrated over all velocities, this gives the density as a function of position. Integrated over all positions, this gives the velocity distribution. Integrated over both position and velocity, it gives the total number of particles in the system. When $f(\mathbf{r}, \mathbf{v})$ is normalized to this total number, it becomes a probability distribution. Then $f(\mathbf{r}, \mathbf{v}) d\mathbf{r} d\mathbf{v}$ is the probability that a particle is in the differential volume $d\mathbf{r}$ around \mathbf{r} and has a velocity in the range $d\mathbf{v}$ around \mathbf{v} .

Other types of distribution functions are also useful. For example, the two-particle distribution $f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2)$ describes the probability that of any two particles one is in the volume $d\mathbf{r}_1$ with velocity in the range $d\mathbf{v}_1$ and the other is simultaneously in $d\mathbf{r}_2$ with velocity in $d\mathbf{v}_2$. This two-particle distribution may be written as the product of two single-particle distributions plus terms representing any correlations that may be present between particles in the two volumes or the two velocity ranges. Similarly, a hierarchy of three-, four-, etc particle distributions and correlations may be built up. Eventually it will reach the N -particle distribution which gives a complete description of the positions and velocities of all N particles in the system. This would then be equivalent to specifying the

position and velocity of every particle—the most detailed description possible. A related form of distribution function, $f(N)$, concentrates on cells rather than particles and specifies the probability that a cell of given size and shape contains N particles.

All these distribution functions may generally have anisotropic dependences on position and velocity and will evolve with time unless they are in a stationary equilibrium state. Anisotropic distribution functions and their properties have been widely explored in recent years. Their complexity provides useful descriptions of stellar motions in both spiral and elliptical galaxies.

The simplest evolution for the one-particle distribution function $f(\mathbf{r}, \mathbf{v}, t)$ occurs in collisionless systems and follows the collisionless Boltzmann equation. Here collisionless means that particle orbits proceed smoothly under the influence of just the mean field. There are no scatterings by other particles or by local fluctuations. Under these conditions $f(\mathbf{r}, \mathbf{v}, t)$ satisfies an equation of continuity. This is equivalent to its total time derivative, following the motion through a 6-dimensional position–velocity phase space, being zero:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \dot{\mathbf{v}} \cdot \nabla_{\mathbf{v}} f = 0. \quad (15)$$

Equation (15) is the collisionless Boltzmann equation—one of the most useful descriptions of stellar dynamics. Notice that the gradients in position and in velocity space are treated equivalently. Although, written this way, it looks like a fairly simple linear partial differential equation for $f(\mathbf{r}, \mathbf{v}, t)$, it is really a complicated non-linear differential-integral equation. This is because the gravitation force, proportional to the acceleration $\dot{\mathbf{v}} = \nabla\phi$, is a function of the density which is itself an integral of $f(\mathbf{r}, \mathbf{v}, t)$ over velocity space. This relation is obtained from Poisson's equation, $\nabla^2\phi(\mathbf{r}, t) = -4\pi G\rho(\mathbf{r}, t)$, and together with the relevant initial and boundary conditions leads to self-consistent solutions.

Very few solutions are known in closed analytic form; most are results of numerical integrations or perturbation theory. The best-known solution occurs for an idealized homogeneous, isotropic, uncorrelated equilibrium distribution. It is the Maxwell–Boltzmann distribution with a Gaussian distribution of velocities. Solutions of equation (15) also provide a zero-order approximation for systems of particles such as globular clusters and rich clusters of galaxies. Their corresponding spatial density is spherically symmetric, decreasing with radius. Such isothermal spheres have to be truncated to describe clusters of finite mass. Truncation by tidal cutoffs, evaporation, inflow, etc for realistic systems will modify their distribution functions.

Spatial and velocity moments of the collisionless Boltzmann equation, similar to those leading to the virial theorem and its generalizations described earlier, yield the Jeans equations of 'stellar hydrodynamics'. These are analogous to the usual fluid equations

of hydrodynamics. However, because short-range atomic interactions dominate fluids, it is a much better approximation to truncate the moment equations at low order for fluids than it is in the stellar dynamical case. Moreover, the general stellar hydrodynamical equations are anisotropic in their spatial and velocity coordinates.

Generalizations of the collisionless Boltzmann equation have been developed to incorporate local fluctuations, such as those described earlier, which lead to dynamical relaxation, evaporation and other instabilities. The most useful of these is the Fokker–Planck description which represents the total gravitational field as the sum of two parts: the smooth average long-range field and the local fluctuating field due to near-neighbor particles. These fluctuations cause the particles to diffuse in velocity space as well as in configuration space. They also incorporate the dynamical friction effects, mentioned earlier, which prevent excessive velocities from being reached. The Fokker–Planck equation provides a more accurate account of processes such as evaporation, core collapse and oscillations which can occur in globular clusters.

The most general and rigorous description of stellar dynamics, and therefore the least solvable, is known as the BBGKY hierarchy (after the initials of its early developers in other subjects, Born, Bogoliubov, Green, Kirkwood and Yvon). It starts from Liouville's equation which is an equation of continuity similar to equation (15) but in a $6N$ -dimensional (Gibbs) phase space rather than in the 6-dimensional (Boltzmann) phase space. Here N is the number of physical particles in the system and each point in the Gibbs phase space of $3N$ position plus $3N$ velocity dimensions represents the state of the entire system of $6N$ particles at a given time. As the system evolves, its representative point moves continuously and smoothly through this phase space because there are no external forces outside the system to perturb it. Thus the Liouville continuity equation is exact. It is a first-order linear differential equation. The problem is that it has $6N$ variables, plus time. In fact, it is just a condensed way of representing the orbits of all the particles in the system.

To make this description useful, it is necessary to successively integrate out all but $6(N-1)$, $6(N-2)$, \dots , 6 of the $6N$ variables in the full distribution function. This leaves a set of N coupled non-linear integro-differential equations for all the N , $N-1$, $N-2$, \dots , one-particle distribution functions—the BBGKY hierarchy. The entire set is equivalent to the equations of motion for all N particles. Truncating the lowest-order equation for $f(\mathbf{r}, \mathbf{v}, t)$ by neglecting its coupling to $f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2, t)$ is equivalent to the collisionless Boltzmann equation. Retaining this coupling, but neglecting any higher-order coupling, gives essentially the Fokker–Planck equation. This formalism has been much studied and gives great insight into the physical nature and relations among different stellar dynamical descriptions. It has found important applications in understanding the linear growth of two- and three-particle correlation functions in galaxy clustering, as well as for plasma physics and the theory of imperfect fluids.

Thermodynamic descriptions

Particle and distribution function descriptions are mathematically complex because they contain much detailed microscopic information about a stellar dynamical system. Thermodynamic descriptions are mathematically much simpler because they deal mainly with macroscopic information which averages over the microscopic detail. This relative simplicity has provided considerable physical insight into the behavior of stellar dynamical systems.

Since the macroscopic properties generally change with time (e.g. even in virialized clusters, particles slowly evaporate), stellar dynamical systems cannot be in equilibrium. Therefore it might be thought that thermodynamics, which is primarily an equilibrium theory, could not apply. Often, however, the timescales for departures from equilibrium are very long, and singular states are essentially unattainable over shorter times. As a result, thermodynamics provides a good approximation for the periods of interest. The system may be in quasi-equilibrium, during which it evolves through a sequence of equilibrium states. In simple systems, each state can be described well in terms of average macroscopic thermodynamic variables such as temperature, pressure, chemical potential, internal energy, volume, total number of particles and entropy, but these quantities change on a timescale which is slow compared with the timescale for a local microscopic configuration of particle positions and velocities to change.

Symmetry properties of different physical systems strongly influence their thermodynamic descriptions. For example, infinite statistically homogeneous systems—which may represent galaxy clustering—have rotational and translational symmetry everywhere. However, finite spherical systems have rotational symmetry only at their center and translational symmetry nowhere. Consequently these infinite systems are described by a grand canonical ensemble in which energy and particles can move across boundaries. Finite, isolated clusters, on the other hand, are described either by a microcanonical ensemble with no transport across boundaries or by a canonical ensemble with only energy transport. (An ensemble is a collection of systems with the same average macroscopic properties but different microscopic configurations consistent with the macroscopic averages.) This leads to differences in their distribution functions and fluctuation spectra.

Thermodynamic behavior in systems dominated by gravity is often ‘counterintuitive’, although really just in the sense that they behave differently from more familiar systems of particles without long-range, unshielded, attractive forces. For example, if one removes energy from a self-gravitating cluster of stars, it becomes hotter. Adding energy makes it cooler. Thus it has a negative specific heat. In fact, this follows simply from the virial theorem for a bound system. Slowly adding energy makes the system’s gravitational well less negative. To maintain their changing quasi-equilibrium state, the particle velocities must decrease on average, and the

system grows cooler. It is essentially the same effect as adding energy to a satellite orbit around a massive body, and so it should be, since the virial theorem applies to both cases.

Applied to finite, isolated spherical clusters of stars, gravitational thermodynamics has been especially useful in elucidating their global instabilities. The isothermal sphere, where all particles have the same mass and temperature, provides a relatively simple illustration. Its density is obtained by solving the collisionless Boltzmann equation and Poisson’s equation for these conditions, or, more easily but less accurately, by solving the gravitational hydrostatic equations with an isothermal gas equation of state. It has a central core, and density $\rho \propto r^{-2}$ in the outer parts, and is usually truncated to keep its mass finite. (Other stellar clusters with more general equations of state similar to polytropic stars can also be manufactured to provide simple models.)

The stability of such an isothermal sphere depends on the relation between its total energy and its total entropy. Systems of the same total energy may have different entropies depending on their size and internal distribution. If the entropy has a local maximum for a given energy, then that configuration is stable to small fluctuations (but possibly metastable to large changes). If the entropy does not have a local maximum, then the system of a given energy can redistribute itself internally to increase its entropy, and the situation is unstable. Such analysis, originally done by Antonov, for an isothermal sphere confined to a spherical box shows that, if the ratio of the central to the boundary density exceeds 709, the system becomes unstable. It tends to evolve away from the isothermal sphere density distribution into a denser central core surrounded by a much less dense halo. Similar core-halo evolution is also found in numerical simulations. Its underlying dynamical mechanism is the slow evaporation of stars which have accumulated just enough energy to escape by their interactions with the fluctuating gravitational field of neighboring stars. Since the total gravitational energy of the isolated system is conserved, its core becomes denser with its energy more negative in order to compensate the positive total energy of the escaping stars.

An infinite statistically homogeneous system provides a contrasting application of gravitational thermodynamics. The cosmological many-body problem is an example. In its simplest case, we start with a uniform random (Poisson) distribution of identical point masses throughout the universe and ask how this distribution changes as the universe expands. In the usual Einstein–Friedmann cosmological models, the cosmological expansion exactly compensates the smooth long-range component of the gravitational field. This leaves only the relatively local fluctuations caused by the discreteness of the particle gravitational fields and by any clustering. These fluctuations can be described by an equation of state which incorporates the gravitational interaction into thermodynamic quantities such as the internal energy and pressure.

Because the universe generally expands more slowly than the timescale for particle configurations within local fluctuations to change, it undergoes a quasi-equilibrium evolution. At any time an equilibrium thermodynamic state provides a good description of the clustering. Its thermodynamic quantities change slowly as the universe expands adiabatically.

Infinite statistically homogeneous systems have a uniform density when averaged over sufficiently large scales or over an ensemble of smaller scales. These infinite systems are characterized by the fluctuations over various scales of macroscopic quantities around their averages. The fluctuations most closely related to observations are the particle distribution functions $f(N, V)$ which give the probability for finding N particles (e.g. galaxies) in a randomly placed volume of size V . Applying thermodynamic fluctuation theory to the cosmological many-body equation of state gives a relatively simple formula for this distribution:

$$f(N, V) = \frac{\bar{N}(1-b)}{N!} [\bar{N}(1-b) + Nb]^{N-1} e^{-\bar{N}(1-b)-Nb}. \quad (16)$$

Here $\bar{N} = \bar{n}V$ is the average number in the volume V for the average number density \bar{n} . The quantity $b = -W/2K$ is the ratio of the gravitational correlation energy to twice the kinetic energy of random motions, averaged over all volumes of size V having a particular shape. The correlation energy, W , is the integral of the r^{-1} interparticle gravitational potential multiplied by the two-particle correlation function $\xi(r)$ over the volume. $\xi(r)$ is the average excess over the random Poisson probability for finding a particle in a differential volume element at a distance r from another particle.

In a completely uncorrelated initial Poisson distribution, $\xi(r) = 0$, $W = 0$ and consequently $b = 0$. In this limit, the distribution function of equation (16) does indeed reduce to the standard Poisson form. As the system evolves, regions where near-neighbor points happen to be closer than average cluster as a result of their enhanced gravity. These clusters subsequently cluster themselves and a hierarchy of non-linear clustering, with a wide range of amplitudes and scales, represented by the increasing value of b , gradually builds up. In the Einstein–Friedmann universe, this non-linear evolution of b can be calculated analytically and it asymptotically approaches unity as the universe expands. This asymptotic limit represents bound clustering on all scales, and is strictly reached only for $\Omega_0 = 1$.

A velocity distribution function can also be derived from equation (16). Compared with the Maxwell–Boltzmann distribution for finite isothermal spheres, the velocity distribution function for the cosmological case is much broader. This is because it includes all levels of clustering, from isolated field galaxies to the richest bound clusters. Numerical computer simulations of the cosmological many-body system verify equation (16) as well as its associated velocity distribution function. Observations of galaxies on the sky and in three

dimensions, using COUNTS IN CELLS as well as VOID and near-neighbor statistics, are also in very good agreement with equation (16). The currently observed value of b is about 0.75.

The BBGKY kinetic hierarchy, mentioned earlier, has also been used to examine hierarchical galaxy clustering. It is partially solvable for the two- and three-particle correlation functions in the linear regime. However, it generally contains less usable observational information than the distribution functions. There are also many more complicated models of galaxy clustering, involving hot and cold dark matter, various forms of initial density and velocity perturbations, biases between the galaxies and dark matter, etc, but their detailed applicability to our universe remains unclear.

Many applications of these main physical descriptions of stellar dynamics—particle orbits, kinetic theory, distribution functions and thermodynamics—have developed over the last century. They have helped provide an understanding of the formation, relaxation and dynamical evolution of star clusters within galaxies. They help explain the streaming motions and arms of spiral galaxies, as well as the triaxial shapes of elliptical galaxies. Combined with the possible existence and effects of massive black holes in galactic nuclei, stellar dynamics helps describe the evolution of the nuclei and the feeding of their black holes. On a larger scale, stellar dynamics gives a background for understanding the formation, relaxation and evolution of galaxy clusters, incorporating effects of their internal dark matter. Even more generally, the principles of stellar dynamics play an important role in accounting for the structure of matter over the largest scales in the universe.

Although stellar dynamics is an old and well-established subject, it is often rejuvenated by new insights and new applications. Some interesting areas for future further developments include the detailed links between dynamics and thermodynamics, implications for galaxy clustering, galaxy and cluster merging, distributions around black holes in galactic nuclei, counterstreaming in galaxies and the development of stellar motions in large star-forming regions.

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