1. Derive (in a Newtonian approximation) the Friedmann equation for a universe with a non-relativistic matter only (no radiation or cosmological constant). [3/12 points]

Imagine a homogeneous sphere of density $\rho$. The force of gravity on the surface of this sphere will yield an acceleration on a matter element so that

$$\ddot{r} = -\frac{GM}{r^2}.$$  

We could multiply both sides by $\dot{r}$ and integrate to find that the sum of the kinetic and potential energies is constant:

$$\frac{1}{2}(\dot{r})^2 - \frac{GM}{r} = U.$$  

Now, we know that for this sphere

$$M = \frac{4}{3}\pi r^3 \rho,$$

where $M$, the total mass of the sphere, is constant. This can be substituted into our last equation to yield

$$\frac{1}{2}(\dot{r})^2 = \frac{4}{3}\pi G\rho r^2 + U.$$  

Rearranging and writing in terms of the scale factor $a$ and the initial radius of the sphere, $r_0$, we obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho + \frac{U'}{a^2 r_0^2}.$$  

2. Now assume that the matter density is exactly equal to the critical value (aka the Einstein – de Sitter model).

a. Derive the age (not the look-back time!) vs. redshift relation in this universe? [2/12 points]

An Einstein-de Sitter universe is described by the parameters $\Omega_{0,\Lambda} = 0$, $\Omega_{0,m} = 1$, and $\kappa = 0$. This is a spatially flat universe with a matter density equal to the critical density. It does not exhibit accelerated expansion, due to the lack of a dark energy component to the total energy density of the universe.

Therefore, the Friedmann equation (including the effects of general relativity) simplifies to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_m$$
Note that from Hubble’s law, \( v = Hr \), we can express the Hubble parameter in terms of the scale factor and its time derivative, \( H(t) = \dot{a}/a \). In addition, recall that the matter density of the universe evolves with time, \( \rho(t) = \rho_m,0 a^{-3} \). Using this information, and defining the Hubble constant as \( H_0^2 = 8\pi G \rho_m,0 / 3 \), we can write the Friedmann equation in its parametric form,

\[
H(z) = H_0 \sqrt{\Omega_{0,m}(1 + z)^3}
\]

where we have used the relationship \( a = 1/(1 + z) \) to express the Friedmann equation in terms of the redshift \( z \).

To derive a relationship between the age of the universe and the redshift, we note that \( dt = (dt/da)(da/dz)dz \), where \( da/dt = \dot{a} = aH \) and \( da/dz = -1/(1 + z)^2 \). Therefore we have

\[
t(z) = \int_0^t dt' = \int_z^\infty \frac{dz'}{H_0 E(z')(1 + z')},
\]
where in the case of an EdS universe \( E(z) = \sqrt{(1 + z)^3} \). Integrating, we obtain,

\[
t(z) = \frac{2}{3H_0} \left[ \frac{1}{(1 + z)^{3/2}} \right]
\]

b. What is its present age in the units of \( t_H = 1 / H_0 \)? [1/12 points]

Using the age vs. redshift relation we derived, we can find the current age of the universe, which corresponds to \( z = 0 \). Plugging in this value gives, in units of the Hubble time,

\[
t(z = 0) = \frac{2}{3H_0} \approx \frac{2}{3} t_H
\]

c. Derive the equation for the distance as a function of redshift for this universe. [2/12 points]

To derive the comoving distance as a function of redshift, we start from the expression

\[
r(t) = c \int_t^{t_0} \frac{dt'}{a(t')},
\]

Changing variables from \( t \) to \( z \) as we did in part a.), we have

\[
r(z) = \frac{c}{H_0} \int_0^z dz \left( \frac{1}{E(z')} \right),
\]

which gives for an EdS universe,

\[
r(z) = \frac{2c}{H_0} \left[ 1 - (1 + z)^{-1/2} \right]
\]

What is the distance to \( z = 1 \) in units of \( D_H = c / H_0 \)? [1/12 points]

To find the comoving distance to \( z = 1 \), we simply plug in the value for the redshift to obtain,

\[
r(z = 1) = (2 - \sqrt{2}) D_H
\]
3. Assume that we live in a universe with $h = 0.7$, $\Omega_m \approx 0.3$, $\Omega_\Lambda \approx 0.7$, and $T_{CMB} = 2.7$ K. Look up the value of the critical density.

a.) Estimate the $\Omega_r$ for the CMB today, using the blackbody energy density formula (look it up). [1/12 points]

The blackbody energy density is given by $u = \alpha T^4$, where $T$ is the temperature that corresponds to the blackbody peak and $\alpha$ is the radiation constant. This relationship can be derived from the integrated blackbody flux, $u = \frac{4\pi B}{c}$. Also recall that the critical density is $\rho_{crit} = 9.21 \times 10^{-30}$ g cm$^{-3}$.

First, we solve for the energy density the cosmic microwave background, $u_{CMB} = \alpha \frac{4\pi B}{c} \approx 4 \times 10^{-13}$ erg cm$^{-3}$.

Given that matter and energy are equivalent, or $u_{CMB} = \rho_{CMB}c^2$, we can express the energy density of the CMB as a mass density. Consequently, we can solve for the density parameter of the CMB at present day,

$$\Omega_{CMB} = \frac{\rho_{CMB}}{\rho_{crit}} = \frac{u_{CMB}}{\rho_{crit}c^2} = 4.8 \times 10^{-5}$$

b.) Estimate the redshift of the transition from the radiation-dominated to the matter-dominated universe (i.e., when $\Omega_m \approx \Omega_r$). [1/12 points]

As stated in the problem, the redshift of transition occurs when the energy density of the universe in radiation (where the universe is radiation dominated at early times) is approximately equal to that of matter.

$$\Omega_r \sim \Omega_m \implies \Omega_{0,r} a^{-3} \sim \Omega_{0,m} a^{-4}$$

$$a \sim \frac{\Omega_{0,r}}{\Omega_{0,m}}$$

Keeping in mind that $a \sim z^{-1}$ for large values of redshift, and approximating $\Omega_{0,r}$ by $\Omega_{0,CMB}$ (since the cosmic microwave background is the dominant component of the radiation energy density), we have

$$z \sim 6250$$

c. Estimate the redshift of the transition from the matter-dominated to the dark energy dominated universe (i.e., when $\Omega_m \approx \Omega_\Lambda$). [1/12 points]

Using the same method as in part b.), we have the following relationship at the time of transition, where $\Omega_\Lambda$ is constant with the expansion of the universe:

$$\Omega_{0,m}a^{-3} \sim \Omega_{0,\Lambda}$$

This implies

$$z \sim \left(\frac{\Omega_{0,\Lambda}}{\Omega_{0,m}}\right)^{1/3} - 1$$

$$z \sim 0.3$$