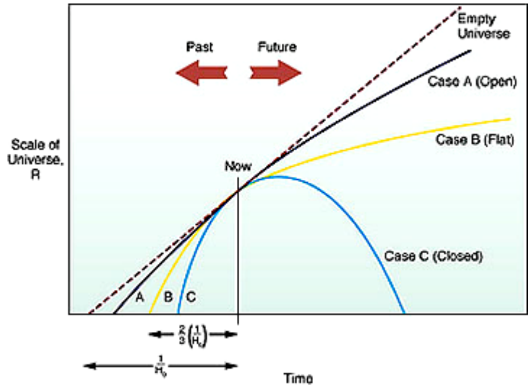
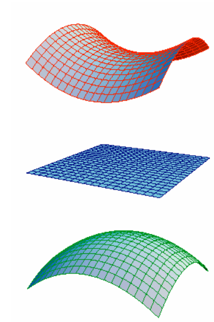
Ay 21 - Lecture 3

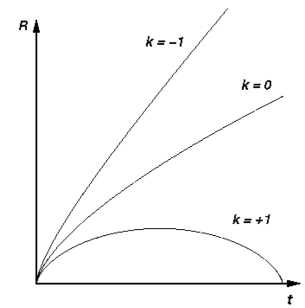
# Global Geometry and Dynamics of the Universe, Part II




## An Overview:



- Cosmological Equations of State
- Computing cosmological models
  - Various examples
- Distances in cosmology
- Basics of cosmological tests
- Horizons




Recall the definitions of the cosmological parameters:

$$\Omega_M = \rho_0/\rho_c = \frac{8\pi G}{3H_0}\rho_0$$

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$$

$$\Omega_M + \Omega_\Lambda + \Omega_k = 1 \qquad \Omega_k = -\frac{k}{R_0^2 H_0^2}$$

and if $\Omega k = 0$, then: $$q_0 = \frac{1}{2}\Omega_M - \Omega_\Lambda$$

**The Friedmann Eqn.** is now:

$$\left(\frac{\dot{R}}{\dot{R}_0}\right)^2 = \Omega_M\left(\frac{R_0}{R}\right) + \Omega_k + \Omega_\Lambda\left(\frac{R}{R_0}\right)^2$$

## Solving The Friedmann Equation

$$\boxed{\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda = -\frac{kc^2}{R^2}}$$

In order to solve it, we also need to define the behavior of mass/energy density $\rho(R)$ of any given mass/energy component, which is generally expressed through the **equation of state**, often written as a relation between pressure and density:

$$P = w\,\rho$$

## The EOS Parameter *W*

- Defined by the equation $p = w\,\rho$
- Often called by itself the "equation of state"
  - Note: this is not necessarily the best way to describe the matter/energy density; it implies a fluid of some kind… This may be OK for the matter and radiation we know, but maybe it is not an optimal description for the dark energy
- Special values:

  $w = 0$ means $p = 0$, e.g., non-relativistic matter

  $w = 1/3$ is radiation or relativistic matter

  $w = -1$ looks just like a cosmological constant

  … but it can have in principle any value, and it can be changing in redshift

## Equation of State (EOS)

Some simple EOS we can consider:

- "Matter", "dust", "galaxies" $P = 0$ (1)
- Radiation $P = \dfrac{\rho c^2}{3}$ (2)
- "Cosmological Constant" $P = -\rho c^2$ (3)

In reality, the universe contains a mix of these components, and maybe others as well…

Each will lead to a different evolution in redshift, and recall the basic GR paradigm:

Density determines the expansion

Expansion changes the density

## Using the EOS

How does the matter/energy density change as the universe expands?

Start with the 3 cases we just described:

$P = 0$ (1)

$P = \dfrac{\rho c^2}{3}$ (2)

$P = -\rho c^2$ (3)

We need to put these into the Fluid equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0$$

Fluid Equation

We get:

(1) $\dot{\rho} + 3\dfrac{\dot{a}}{a}\rho = 0 \;\Rightarrow\; \dfrac{1}{a^3}\dfrac{\partial}{\partial t}\left(\rho a^3\right) = 0 \;\Rightarrow\; \rho \propto a^{-3}$ (4)

(2) $\dot{\rho} + 4\dfrac{\dot{a}}{a}\rho = 0 \;\Rightarrow\; \dfrac{1}{a^4}\dfrac{\partial}{\partial t}\left(\rho a^4\right) = 0 \;\Rightarrow\; \rho \propto a^{-4}$ (5)

(3) $\dot{\rho} = 0 \;\Rightarrow\; \rho$ const (6)

## Evolution of the Density

Generally, $\rho \sim R^{-3(w+1)}$

- Matter dominated ($w = 0$): $\rho \sim R^{-3}$
- Radiation dominated ($w = 1/3$): $\rho \sim R^{-4}$
- Cosmological constant ($w = -1$): $\rho = constant$
- Dark energy with $w < -1$ e.g., $w = -2$: $\rho \sim R^{+3}$
  - Energy density *increases* as is stretched out!
  - Eventually would dominate over even the energies holding atoms together! ("Big Rip")

In a mixed universe, different components will dominate the global dynamics at different times

Note also that in principle, $w$ can be a function of time, density…

# Specific Models

**Consider several simple models:**

- $\boldsymbol{k} = 0$, matter dominated, Einstein de Sitter
- $\boldsymbol{k} = 0$, radiation dominated
- $\boldsymbol{k} < 0$, $\rho = 0$, Milne Model
- $\boldsymbol{k} < 0$, $\rho > 0$
- $\boldsymbol{k} > 0$
- $\Lambda$ dominated

# $\boldsymbol{k} = 0$, matter dominated Einstein de Sitter

Friedman Equation

with $\boldsymbol{k} = 0$ $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho_0 a^{-3}$$

$$\Rightarrow \dot{a}^2 = \frac{8}{3}\pi G\rho_0 a^{-1}$$

$$\Rightarrow \frac{\partial a}{\partial t} = \pm\sqrt{\frac{8}{3}\pi G\rho_0}\, a^{-1/2}$$

$$\Rightarrow \int a^{1/2} da = \pm\sqrt{\frac{8}{3}\pi G\rho_0} \int dt$$

$$\Rightarrow a^{3/2} = \pm 2\sqrt{\frac{8}{3}\pi G\rho_0}\; t$$

$$\Rightarrow a = \pm 2\left(\frac{8}{3}\pi G\rho_0\right)^{1/3} t^{2/3}$$

i.e. $\boxed{a \propto t^{2/3}}$

# $\boldsymbol{k} = 0$, radiation dominated

Friedman Equation

with $\boldsymbol{k} = 0$ $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho_0 a^{-4}$$

$$\Rightarrow \dot{a} \propto a^{-1}$$

$$\Rightarrow \int a\,da \propto t$$

$$\Rightarrow a^2 \propto t$$

$$\boxed{\Rightarrow a \propto t^{1/2}}$$

# $\boldsymbol{k} = < 0$, ρ = 0 Milne Model

Friedman Equation

with $\boldsymbol{\rho} = 0$ $\left(\frac{\dot{a}}{a}\right)^2 = -\frac{kc^2}{a^2}$

since $\boldsymbol{k} < 0$ $\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 \propto \frac{+1}{a^2}$

$$\Rightarrow \dot{a} = \text{const}$$

$$\boxed{\Rightarrow a \propto \pm t}$$

# $\boldsymbol{k} < 0$, ρ>0

Friedman Equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho - \frac{kc^2}{a^2}$$

$$\dot{a}^2 > 0$$

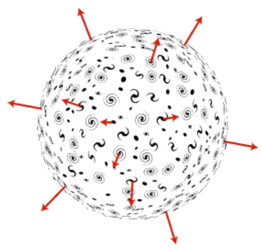
$$t \to \infty \quad a \to \infty,\, a^{-1} \to 0$$

$$\frac{1}{a^2} >> \frac{1}{a^3} >> \frac{1}{a^4}$$

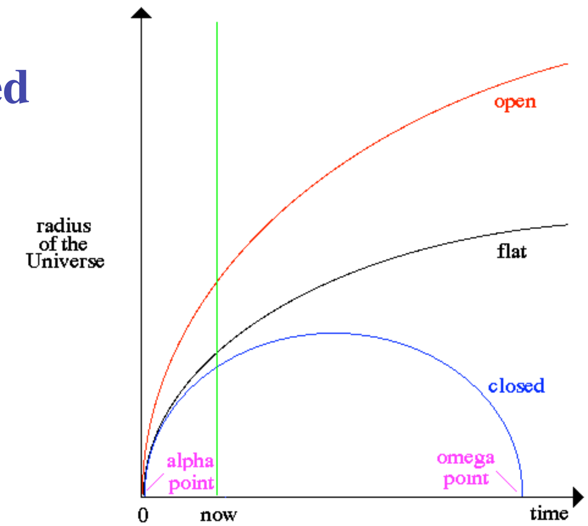
either matter or radiation dominated models tend to Milne model $\boxed{\Rightarrow a \propto \pm t}$

# Milne Cosmology

- Galaxies are created at one point in a flat spacetime
- Galaxies behave as massless test particles
  - then “trivially” get $v = H_0 d$ (faster particles move further)
- Special relativity holds
  - Lorentz contraction lets us have infinitely many galaxies (almost all right at the edge of our expanding bubble)
- No special locations
  - Milne was the first to state this “Cosmological Principle”



# Matter dominated model

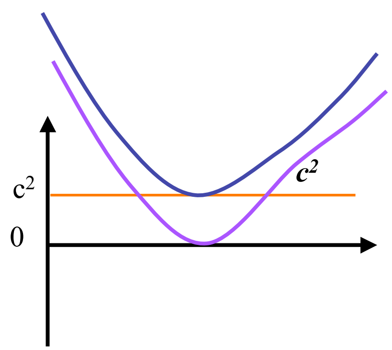



# $k$ = +1, P>0

(3) Friedman Equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho - \frac{kc^2}{a^2}$$

$$\dot{a}^2 = \frac{8}{3}\pi G\rho a^2 - c^2$$




if $\rho \propto a^{-3}$ or $\rho \propto a^{-4}$ then $\rho a^2 \propto a^{-n}$ $(n = 1,2)$ decreases

at some point $\dot{a}^2 = 0$ and since

(8) $$\boxed{\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G\left(\rho + 3\frac{P}{c^2}\right)}$$

Acceleration Equation

$\ddot{a}^2 < 0$ Therefore, collapse is inevitable

# $\Lambda$ dominated

(3) Friedman Equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho_\Lambda - \frac{kc^2}{a^2}$$

$$\dot{a}^2 = C_0\rho a^2 - kc^2$$

assuming $a$ is allowed to grow then eventually $C_0 a^2$

dominates over $-kc^2$ no matter what value of $\boldsymbol{k}$

$\rho_\Lambda > 0$ Universe expands for ever

# Models with both matter & radiation

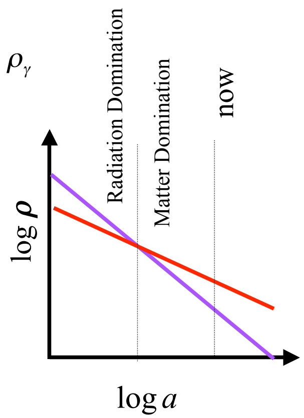
Fluid equation is modified with $\rho \rightarrow \rho_{\mathrm{m}} + \rho_{\gamma}$

$$\Rightarrow \frac{1}{a^3}\frac{\partial}{\partial t}\left(\rho_{\mathrm{m}} a^3\right) + \frac{1}{a^4}\frac{\partial}{\partial t}\left(\rho_{\gamma} a^4\right) = 0$$

Assuming negligible conversion of mass to radiation,

both terms must separately be zero

4 $\rho_{\mathrm{m}} \propto a^{-3}$ 5 $\rho_{\gamma} \propto a^{-4}$

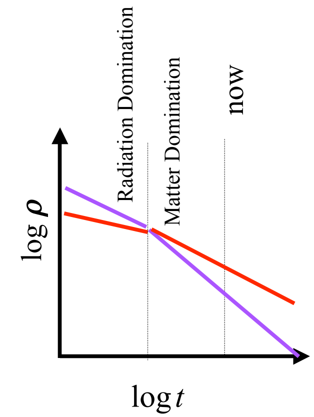



# Models with both matter & radiation

Harder to solve for ***ρ (t)***

However we can assume that $\boldsymbol{k} = 0$ and either Radiation or matter dominate

| | | γ-dom | m-dom |
|---|---|---|---|
| | $a(t)$ | $\propto t^{1/2}$ | $\propto t^{2/3}$ |
| 4 | $\rho_{\mathrm{m}} \propto a^{-3}$ | $\propto t^{-3/2}$ | $\propto t^{-2}$ |
| 5 | $\rho_{\gamma} \propto a^{-4}$ | $\propto t^{-2}$ | $\propto t^{-8/3}$ |




Generally, $\dfrac{8\pi G\rho}{3} = H_0^{\,2}\left(\Omega_{\Lambda,0} + \Omega_{\mathrm{m},0}a^{-3} + \Omega_{\gamma,0}a^{-4}\right)$

# What is Dominant When?

Matter dominated ($w = 0$): $\rho \sim R^{-3}$
Radiation dominated ($w = 1/3$): $\rho \sim R^{-4}$
Dark energy ($w \sim -1$): $\rho \sim$ *constant*

- Radiation density decreases the fastest with time
  - Must increase fastest on going back in time
  - Radiation must dominate early in the Universe
- Dark energy with $w \sim -1$ dominates last

Radiation domination → Matter domination → Dark energy domination

# Dynamics of the Universe

$$R(t) \sim t^{\,2/[3(w+1)]}$$

- Matter dominated ($w = 0$): $R \sim t^{\,2/3}$
  - Decelerating
- Radiation dominated ($w = 1/3$): $R \sim t^{\,1/2}$
  - Decelerating
- Cosmological constant ($w = -1$): $R \sim e^{\,\lambda t}$ (special)
  - Accelerating
- Where is the transition?
  - $w > -1/3$ decelerating
  - $w < -1/3$ accelerating

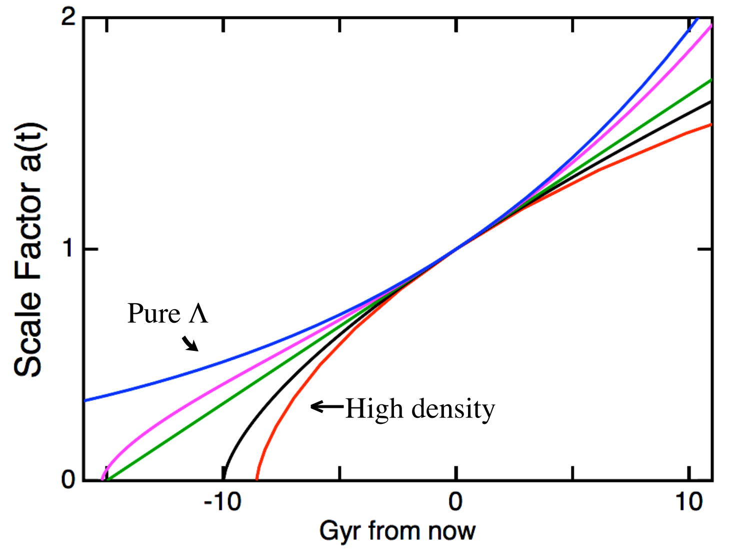



Fig. 10.— Scale factor *vs.* time for 5 different models: from top to bottom having $(\Omega_{mo}, \Omega_{vo}) = (0,1)$ in blue, (0.25,0.75) in magenta, (0,0) in green, (1,0) in black and (2,0) in red. All have $H_o = 65$.

## Distances in Cosmology

A convenient unit is the **Hubble distance,**

$$D_H = c / H_0 = 4.283\ h_{70}^{-1}\ \text{Gpc} = 1.322 \times 10^{28}\ h_{70}^{-1}\ \text{cm}$$

and the corresponding **Hubble time**,

$$t_H = 1 / H_0 = 13.98\ h_{70}^{-1}\ \text{Gyr} = 4.409 \times 10^{17}\ h_{70}^{-1}\ \text{s}$$

At low $z$'s, distance $D \approx z\, D_H$ . But more generally, the comoving distance to a redshift $z$ is:

$$D_{\rm C} = D_{\rm H} \int_0^z \frac{dz'}{E(z')}$$

where

$$E(z) \equiv \sqrt{\Omega_{\rm M}\,(1+z)^3 + \Omega_k\,(1+z)^2 + \Omega_\Lambda}$$

## Distances in Cosmology

But the quantity really useful in computing the various physical quantities of interest is the "transverse comoving distance", where we account for the curvature:

$$D_{\rm M} = \begin{cases} D_{\rm H}\,\frac{1}{\sqrt{\Omega_k}}\,\sinh\left[\sqrt{\Omega_k}\,D_{\rm C}/D_{\rm H}\right] & \text{for } \Omega_k > 0 \\ D_{\rm C} & \text{for } \Omega_k = 0 \\ D_{\rm H}\,\frac{1}{\sqrt{|\Omega_k|}}\,\sin\left[\sqrt{|\Omega_k|}\,D_{\rm C}/D_{\rm H}\right] & \text{for } \Omega_k < 0 \end{cases}$$

where $\Omega_k$ is defined by:

$$\Omega_{\rm M} + \Omega_\Lambda + \Omega_k = 1$$

$$\Omega_{\rm M} \equiv \frac{8\pi\, G\, \rho_0}{3\, H_0^2}$$

$$\Omega_\Lambda \equiv \frac{\Lambda\, c^2}{3\, H_0^2}$$

## Distances in Cosmology

We can derive this for using the RW metric:

$$c^2 dt^2 = R^2 du^2 = R^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right\}$$

To simplify, let's put ourselves at the origin, then the light path is purely radial,
and $d\theta$ and $d\phi = 0$, so:

$$c^2 dt^2 = R^2 \left\{ \frac{dr^2}{1-kr^2} \right\}$$

Taking the square root of both sides and integrating:

$$\int_{t_0}^{t_1} \frac{c}{R} dt = \int_u^0 \frac{dr}{(1-kr^2)^{1/2}}$$

# Distances in Cosmology

In general this is non-analytic. In a special case of a $\Lambda = 0$ Universe, we have $q_0 = \Omega_0 / 2$, and:

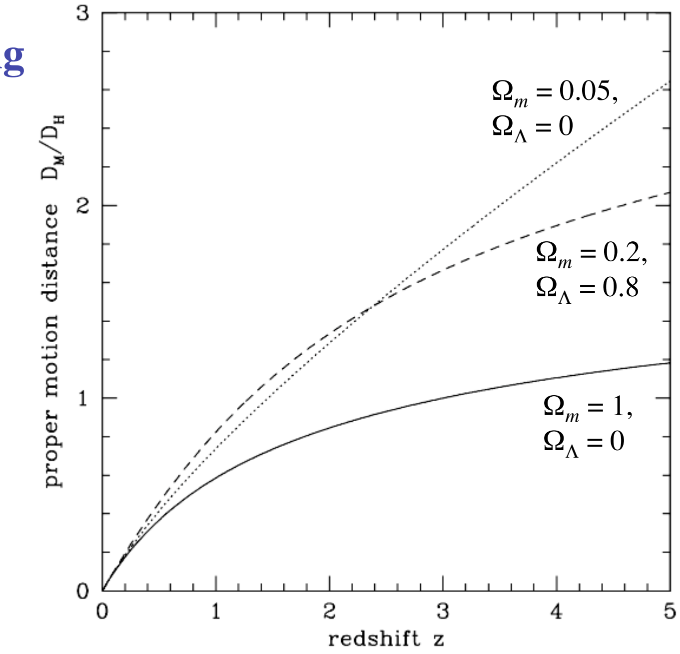
$$d_p = \frac{c}{H_0 q_0^2 (1+z)} \left\{ q_0 z + (q_0 - 1) \left[ (2q_0 z + 1)^{1/2} - 1 \right] \right\}$$

For a non-zero $\Lambda$ universe:

$$d_p = |\Omega_k|^{-\frac{1}{2}} \sinh \left\{ |\Omega_k|^{\frac{1}{2}} \int_0^z \left\{ (1+z)^2 (1 + \Omega_M z) - \Omega_\Lambda z (2+z) \right\}^{\frac{1}{2}} dz \right\}$$

Assuming $\Omega_k<0$, if $\Omega_k>0$ then the *sinh* becomes a *sin* and if $\Omega_k=0$ then the *sinh* and the $\Omega_k$ drop out and all that's left is the integral, which has to be evaluated numerically.

# Comoving Distance




# Luminosity Distance

In relativistic cosmologies, observed flux (bolometric, or in a finite bandpass) is:

$$f = L / [\, (4\pi D^2)\, (1+z)^2 \,]$$

One factor of $(1+z)$ is due to the energy loss of photons, and one is due to the time dialation of the photon rate.

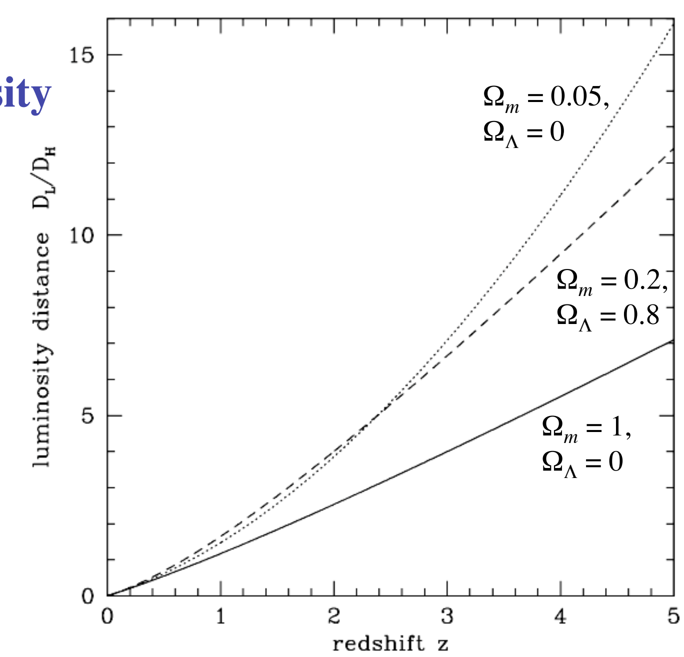
A **luminosity distance** is defined as $D_L = D\,(1+z)$, so that $f = L / (4\pi D_L^2)$.

For a specific flux, however,

$$S_\lambda = \frac{1}{(1+z)} \frac{L_{\lambda/(1+z)}}{L_\lambda} \frac{L_\lambda}{4\pi D_\mathrm{L}^2}$$

(since Angstroms are also stretched by 1+z)

# Luminosity Distance



# Angular Diameter Distance

Angular diameter of an object with a fixed ***comoving*** size $X$ is by definition

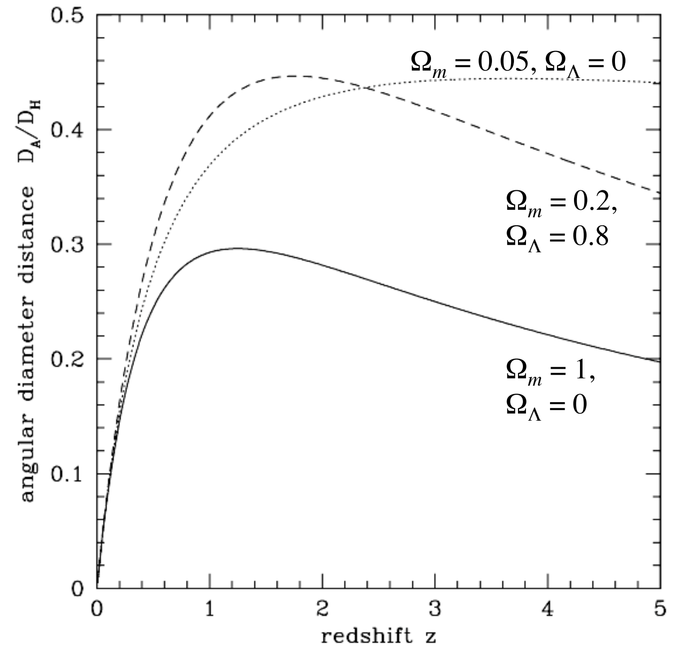
$$\theta = X / D$$

However, an object with a fixed ***proper*** size $X$ is $(1+z)$ times larger than in the comoving coordinates, so its apparent angular diameter will be

$$\theta = (1+z)\, X / D$$

Thus, we define the **angular diameter distance** $D_A = D / (1+z)$ , so that the angular diameter of an object whose size is fixed in proper coordinates is $\theta = X / D_A$

# Angular Diameter Distance




# Volume Element

$$dV_C = D_H \frac{(1+z)^2\, D_A^2}{E(z)}\, d\Omega\, dz$$

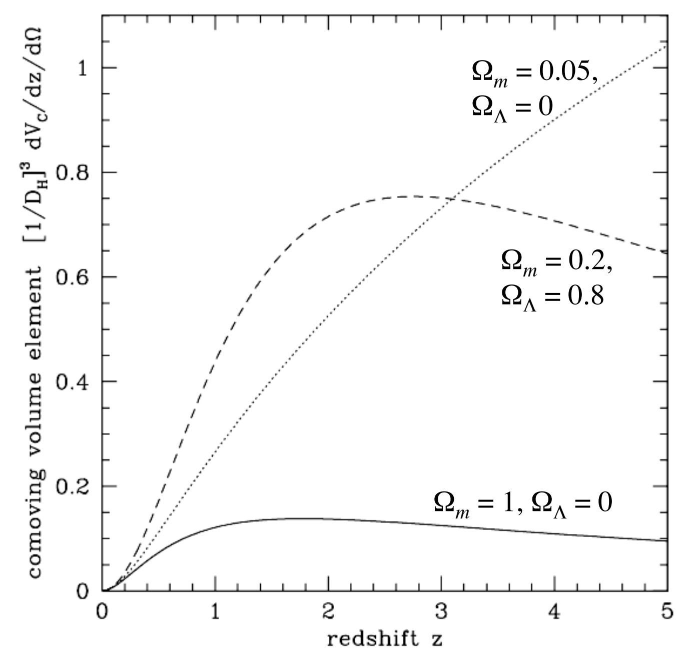
This is useful, e.g., when computing the source counts.

Generally, it has to be evaluated numerically.

The total volume out to some $z$, over the whole sky, is:

$$V_C = \begin{cases} \left(\frac{4\pi\, D_H^3}{2\Omega_k}\right)\left[\frac{D_M}{D_H}\sqrt{1+\Omega_k\,\frac{D_M^2}{D_H^2}} - \frac{1}{\sqrt{|\Omega_k|}}\,\mathrm{arcsinh}\left(\sqrt{|\Omega_k|}\,\frac{D_M}{D_H}\right)\right] & \text{for } \Omega_k > 0 \\ \frac{4\pi}{3}\, D_M^3 & \text{for } \Omega_k = 0 \\ \left(\frac{4\pi\, D_H^3}{2\Omega_k}\right)\left[\frac{D_M}{D_H}\sqrt{1+\Omega_k\,\frac{D_M^2}{D_H^2}} - \frac{1}{\sqrt{|\Omega_k|}}\,\arcsin\left(\sqrt{|\Omega_k|}\,\frac{D_M}{D_H}\right)\right] & \text{for } \Omega_k < 0 \end{cases}$$

# Volume Element



# Age and Lookback Time

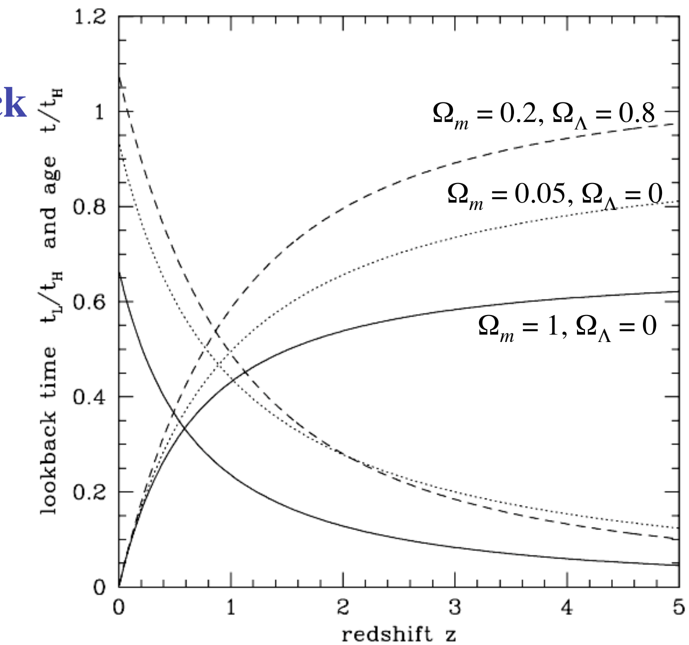
The time elapsed since some redshift $z$ is:

$$t_{\mathrm{L}} = t_{\mathrm{H}} \int_0^z \frac{dz'}{(1+z')\,E(z')}$$

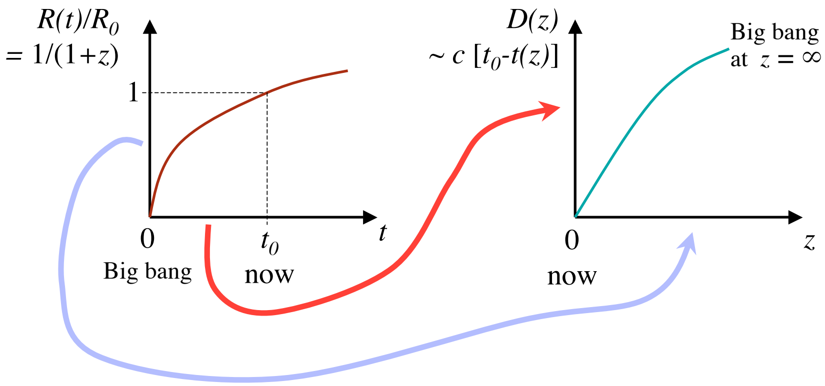
Generally it has to be integrated numerically, except in some special cases, such as $\Lambda = 0$.

Integrating to infinity gives the age of the universe, and the difference of the two is the age at a given redshift.

# Age and Lookback Time

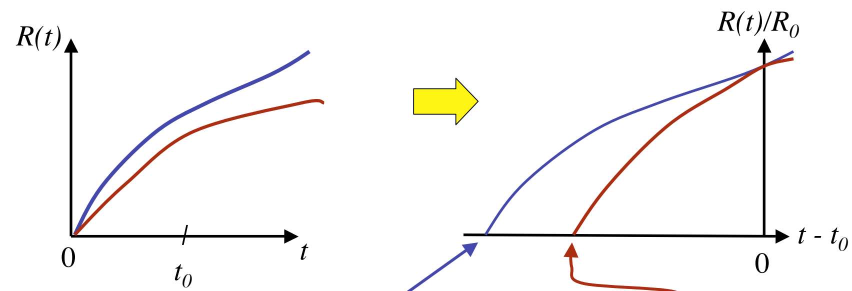



# The Basis of Cosmological Tests




All cosmological tests essentially consist of comparing some measure of (relative) distance (or look-back time) to redshift. Absolute distance scaling is given by the $H_0$.

# Cosmological Tests: Expected Generic Behavior of Various Models




**Models with a lower density and/or positive $\Lambda$ expand faster, are thus larger, older today, have more volume and thus higher source counts, at a given $z$ sources are further away and thus appear fainter and smaller**

**Models with a higher density and lower $\Lambda$ behave exactly the opposite**