Ay 21: Galaxies and Cosmology - Winter 2007	February 21, 2008

Solutions to Midterm Exam

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These solutions are based on those of Christopher Klein.

Problem 1: Definitions [32 points, 4 for each item]

Question: *Define or explain briefly (in a few sentences at most):*

a. The distinction between commoving and proper coordinates?

b. You and an alien astronomer in a galaxy which for you is at $z = 1$ observe a quasar which for you is at $z = 2$ (along the same line of sight); what is the redshift of the quasar from the viewpoint of your alien colleague?

c. List at least 3 distinct methods to measure the H_0, and their principal advantages and disadvantages.

d. What is the K-correction, and what does it depend on?

e. WIMPs and MACHOs?

f. What is the Einstein radius, and the difference between strong and weak lensing?

g. Cosmological constant vs. quintessence, and what is that parameter w?

h. The recombination epoch, its redshift, approx. age of the universe at the time?

Solution:

a: Coordinates

Comoving coordinates, as the name indicates, are those which account for the expansion of the universe. Thus, if two galaxies are a certain comoving distance away, this number is not going to change as the universe expands. The proper distance, on the other hand, which is fixed in time, will increase as the universe expands.

b: Alien redshift

The redshift measures relative sizes of the universe at time of emission. Thus, if

$$1 + z = \frac{a(t_{obs})}{a(t_e)},$$

then

$$\frac{a(t_{QSO})}{a(t_{alien})} = \frac{a(t_{QSO})}{a(t_{now})}\frac{a(t_{now})}{a(t_{alien})} = \frac{1 + z_{QSO}}{1 + z_{alien}}$$

$$\Rightarrow z_{QSO,alien} = \frac{1}{2}.$$

c: Measuring H_0

To measure H_0, we have to correlate redshift and the distance. Thus, the idea is to present ways to measure distance, which require standard candles, in general. Notes from lecture 4 have many methods listed; Cepheids, supernovae and empirical luminosity laws, such as the Tully-Fisher relation, are a few of them.

d: K-correction

When observing galaxies in a sufficiently high redshift, features one would see in a given bandpass are moved into higher wavelength areas of the spectrum. Moreover, $\Delta\lambda_{obs} \neq \Delta\lambda_{em}$. Thus, since bandpass magnitudes give us crucial information on objects, it is necessary to correct for this effect - that is the K-correction.

e: WIMPs and MACHOs

Weakly Interacting Massive Particles and MAssive Compact Halo Objects are two candidates for explaining dark-matter. The first are a class of non-baryonic particles, yet undetected. MACHOs, on the other hand, are astrophysical objects such as brown dwarfs, that do not emit (visible) light.

f: Gravitational Lensing

When light travels around a massive object, its path is bent by the object's gravity. When the foreground object is symmetric and exactly in front of the lensed object, one sees a perfect ring around it, which has radius equal to the Eisntein Radius. This angle also defines whether we see strong lensing - that is, when the line of sight of the lensed object is close enough to the lens so that we see multiple images of it - or weak lensing, in which the object is simply stretched and magnified.

g: Dark energy

The cosmological constant is, as its name says, constant. Quintessence, on the other hand, varies in time and space, i.e. it is also non-homogeneous. They are both possibilities for the observed dark energy component of the universe. The parameter w, also called *equation of state*, relates pressure to energy density:

$$P = w\epsilon.$$

The cosmological constant is assumed to have $w = -1$, that is, negative pressure, which causes the universe to accelerate. Quintessence has $w = w(t, \vec{r})$.

h: Recombination epoch

As the universe expanded, it gradually cooled down. At one point, temperature was low enough so that protons and electrons could combine together in order to form atoms. Since temperature scales with redshift, it is possible to determine that $z_{recomb} \approx 1100$, which corresponds to an age of about 300,000 years.

Problem 2: CMBR Fluctuations [45 points]

Question: *Assume a spatially flat universe, with $\Omega_{total} = \Omega_m = 1$ and that $H_0 = 70$ km/s/Mpc.*

a. Derive the formula for the angular diameter distance $D_A(z)$ in this cosmological model, and for the age of the Universe $t(z)$.[10 points]

b. Compute the proper size in Mpc of the particle horizon diameter at the time of recombination, at $z = 1100$ (hint: from how far would light have traveled at that time in the history of the universe?) [5 points].

b. Compute the corresponding angular size as observed now, if $\Omega_{total} = \Omega_{matter} = 1$ [5 points].

d. Same as (a), but for the empty Universe with $\Omega_{total} = 0$ [5 points].

e. Same as (b), for the empty Universe model. [5 points]

f. Same as (c), for the empty Universe model. [5 points]

d. Comment on these results and the actual observed angular scale ($\sim 0.9\,^\circ$) for the first Doppler peak of the CMBR fluctuations [5 points].

Solution:

a. For a flat Universe the FRW metric is :

$$ds^2 = -c^2dt^2 + a(t)^2[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)]$$

Thus, an object that we see at redshift z, sustaining an angle $\Delta\theta$ has a physical size $S_{phys} = a(z)\chi(z)\Delta\theta \equiv D_A(z)\Delta\theta$. So we have that

$$D_A(z) = \frac{1}{1+z}\chi(z) = \frac{1}{1+z}\frac{c}{H_0}\int_0^z \frac{dz'}{\sqrt{(1+z')^3}}$$

where in the last equality we used the assumption that $\Omega_{tot} = \Omega_m = 1$ Computing the integral, we get :

$$D_A(z) = \frac{2c}{H_0(1+z)}(1 - \frac{1}{\sqrt{1+z}})$$

The age of the Universe at redshift z is given by

$$t(z) = \int_0^{t(z)} dt = \int_0^{a(z)} \frac{da}{aH(a)} = \frac{1}{H_0}\int_z^{+\infty} \frac{dz'}{(1+z')\sqrt{(1+z')^3}} = \frac{2}{3H_0}\frac{1}{(1+z)^{3/2}}$$

b. At $z = 1100$, the universe can be approximated as completely dominated by matter. So, the Einstein-de Sitter universe is a reasonable approximation. There are two ways of answering this question, differing by a factor of 3 depending on how the question was understood :

• The hint suggests that we use the age of the Universe at that time, multiply it by the speed of light to get the total physical distance a photon emitted at the Big Bang has travelled before reaching us. This gives :

$r_{horizon} = ct(z) = \frac{2c}{3H_0}\frac{1}{(1+z)^{3/2}} = 0.078$ Mpc. The horizon diameter is then $d_{hor} = 0.157$ Mpc.

• Another interpretation of the question is : what is the physical size, at redshift z, of the particle horizon at redshift z ? The comoving horizon at redshift z can be found by analogy with the horizon now :

$$\chi_{hor}(z) = \int_0^{t(z)} \frac{cdt}{a(t)} = \frac{c}{H_0}\int_z^{+\infty} \frac{dz'}{\sqrt{(1+z')^3}} = \frac{2c}{H_0}\frac{1}{\sqrt{1+z}}$$

Thus the physical size, at redshift z, of this Horizon is

$$r_{hor} = a(z)\chi_{hor}(z) = \frac{2c}{H_0}\frac{1}{(1+z)^{3/2}} = 0.234 Mpc$$

The physical diameter of the horizon at that time is thus $d_{hor} = 0.468$ Mpc.

Note that this size is three times bigger than the first answer. The reason is the first answer gave the physical distance *travelled* by photons. But as soon as the photon has travelled an infinitesimal distance, the expansion of the Universe makes this distance bigger. Imagine an ant walking on an inflating ball at a constant speed v, for a time t. The ant will have walked a distance $v \times t$, but when it arrives, if someone measures the path it has walked, it will be larger as the sphere has expanded.

c. If one wants to compute the angular size of the horizon, one needs the second result we gave previously for the physical size of the Horizon (and not the distance measured by a travelling photon). By definition of the angular diameter distance, this size is given by

$$\Delta\theta = \frac{d_{hor}}{D_A(z)}$$

Using now that $z = 1100 \gg 1$, one can simplify the expressions given previously :

$$D_A(z) \approx \frac{2c}{H_0 z} \quad , \quad d_{hor} \approx \frac{4c}{H_0 z^{3/2}} \quad , \quad so \;\; \Delta\theta \approx \frac{2}{z^{1/2}} \approx 0.06 \; rad = 3.5 \; deg$$

d. For an empty Universe the Friedman equation is

$$H^2(t) = -\frac{kc^2}{a^2} = H_0^2\frac{a_0^2}{a^2}$$

This can be solved easily to find

$$a(t) = H_0 a_0 t$$

So the age-redshift relation is very simple :

$$t(z) = \frac{1}{H_0(1+z)} = \frac{t_0}{1+z} \quad , \quad t_0 = H_0^{-1}$$

The comoving distance up to a redshift z is also easily integrated :

$$\chi(z) = \int_{t(z)}^{t_0} \frac{cdt}{a(t)} = \frac{c}{H_0 a_0} \ln(1+z)$$

Such a Universe is clearly open, and for simplicity we choose $k = -1$ which imposes $a_0 = \frac{c}{H_0}$ and thus $\chi(z) = \ln(1+z)$. The metric for an open Universe is

$$ds^2 = -c^2 dt^2 + a(t)^2 [d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)]$$

So in that case the angular diameter distance is given by

$$D_A(z) = a(z) \sinh(\chi(z)) = \frac{c}{H_0(1+z)} \sinh(\ln(1+z))$$

e. Here the distinction between the two horizon "sizes" is very important. The first version just gives the total distance travelled by a photon since $t = 0$, which is $ct(z)$. The comoving horizon size, on the other hand, is infinite. In fact an empty Universe has no reason to be curved, and in that case one can show that the metric is flat, after making a change of coordinates. So this empty Universe is just an infinite, static Universe, with an infinite horizon as there is no Big Bang.

f. If we still do as if we did not know that this Universe is in fact flat, and use the distance travelled by photons since t=0, we get $\Delta\theta \approx \frac{2}{z}$ for large z, which corresponds to an angular size of about 0.1 degree.

g. At the time the CMBR was formed (recombination) the universe was mostly dominated by matter and radiation. (Matter-radiation equality took place at $z \approx 3570$.)

In part (c) we calculated the size of the particle Horizon. The first Doppler peak takes place at the *sound horizon*, which is the further distance a sound wave could have travelled at that time. When radiation is important, the sound speed is $c_s = c/3$, so the sound horizon would be one third of the particle horizon in size. Our result in part (c), divided by 3, is thus relatively close to the observationally measured angular scale of $\sim 0.9°$. However, it is slightly larger, as we should expect. Matter causes the angular scale of the first Doppler peak in the CMBR to grow and the cosmological constant causes this angular scale to shrink. So, in our actual universe, we observe the angular scale of the CMBR fluctuations to be $\sim 0.9°$, which is smaller than the $1.15°$ we calculate in part b. We can conclude that the reduced scale of the actual measurements comes about as a result of the cosmological constant.

Problem 3: The Cosmic Neutrino Background [18 points]

Question: *The Cosmic Neutrino Background is expected to have the number density (in neutrinos per cm3) nearly equal to (actually 9/11 of) the number density of photons in CMBR.*

a. Estimate the CMBR photon number density today, assuming TCMBR = 2.7 K, and from that the relict neutrino number density today [5 points].

b. Estimate the number of these relict neutrinos passing through your body every second; state your assumptions [5 points].

c. How massive would these neutrinos have to be (in eV) in order to account for all of the dark matter (state your choices of the relevant cosmological parameters) [8 points].

Solution:

a. The energy density is given by :

$$u(T) = aT^4,$$

where $a = 7.56 \times 10^{-15}$ erg cm^{-3} K^{-4}. Since the temperature of CMBR is approximately 2.7 K, we get that $u(T) = 4.02 \times 10^{-13}$ erg cm^{-3}. Then,

$$n_\gamma \approx \frac{u(T)}{kT} \approx 1080 \text{ cm}^{-3}.$$

So

$$n_\nu = \frac{9}{11} n_\gamma \approx 882 \text{ cm}^{-3}.$$

b. The rate of neutrinos crossing an area S is given by:

$$\Gamma = nSv.$$

A human body is approximately 2 m × 50 cm, and the neutrinos travel almost at light speed, $v \simeq c$. All that combined yields

$$\Gamma = 2.6 \times 10^{17} \text{ s}^{-1}.$$

c. In order to account for all the nonbaryonic mass in the universe, the average neutrino mass would have to be

$$m_\nu c^2 = \frac{\Omega_{dm,0}\ \epsilon_{c,0}}{n_\nu}.$$

Given a density parameter in nonbaryonic dark matter of $\Omega_{dm,0} \approx 0.26$, this implies that a mean neutrino mass of

$$m_\nu c^2 \approx \frac{0.26(5200 \text{ MeV m}^{-3})}{8.82 \times 10^8 \text{ m}^{-3}} \approx 1.5 \text{ eV}$$

would be necessary to account for all of the dark matter.

Problem 4: Galactic halo [40 points]

Question: *Consider a typical disk galaxy like the Milky Way, with a flat rotation curve with Vcirc = 220 km s-1, and a halo extending out to Rmax = 100 kpc. Assume that we live in an Einstein - de Sitter universe with m = 0 = 1, and h = 0.5.*

a. Derive the formula for the free-fall time as a function of the object's mass, and the initial radius [10 points]

b. What is the total mass of this galaxy? [5 points]

c. If it formed via dissipationless collapse, what was the free-fall time? How does it compare with the orbital period at the galaxy's edge today? (hint: what is the ratio of the radius today to the initial radius?) [5 points]

d. What are the present density and the age in this universe? [10 points]

e. Assuming that the halo is virialized today, at what redshift did it start collapsing? [10 points] f. How old was the universe then? [5 points]

Solution:

(a) Consider a particle sitting initially at radius R_i with vanishing velocity. It falls towards the center as it is attracted by the central mass M. Its acceleration is given by

$$\ddot{R} = -\frac{GM}{R^2}$$

After integration this gives

$$\dot{R}^2 = 2GM(\frac{1}{R} - \frac{1}{R_i})$$

Separating variables, integrating from $R = 0$ to $R = R_i$, and finally making the change of variables $u = R/R_i$ in the integral, we find

$$t_{ff} = \sqrt{\frac{R_i^3}{2GM}} \int_0^1 \sqrt{\frac{u}{1-u}} du = \frac{\pi}{2}\sqrt{\frac{R_i^3}{2GM}}$$

(b) The circular velocity is $V_{circ}^2 = \frac{GM(R)}{R}$. So using this formula at the edge of the halo :

$$M = \frac{R_{max} V_{circ^2}}{G} \approx 1.1 \times 10^{12} M_\odot$$

(c) If the galaxy formed via dissipationless collapse, (i.e. the total energy is conserved), then, using the virial theorem gives

$$E = -\frac{GM}{R_i} = -\frac{GM}{2R_f}$$

so

$$R_{max} = R_f = R_i/2$$

The free-fall time is thus given by :

$$t_{ff} = \frac{\pi}{2}\sqrt{\frac{8R_{max}^3}{2GM}} = \pi\sqrt{\frac{R_{max}^3}{GM}}$$

The orbital period at the edge of the galaxy is given by

$$t_{\Pi} = \frac{2\pi R_{max}}{V_{circ}} = 2t_{ff}$$

which gives $t_{\Pi} = 2.8Gyrs$, $t_{ff} = 1.4Gyr$.

(d) Using the Friedmann equation where $\Omega_K = \Omega_\Lambda = 0$, we see that

$$H = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho.$$

Using $H_0 = 50kms^{-1}Mpc^{-1}$, we find $\rho = 4.7 \times 10^{-30}gcm^{-3}$. In an Einstein-de Sitter model, the age of the universe is given by

$$t_0 = \frac{2}{3H_0},$$

which we calculated in the first homework. , Using H_0 above, we find $t_0 = 13.1Gyrs$.

(e) Following the "Spherical Top-Hat" methods in lecture 8, we can sketch the evolution of the overdense region of the universe (our galaxy) as a closed universe using the virial theorem and conclude that:

$$t_{turn} = \frac{1}{2}t_{vir} = 1/2\ t_0 \quad \text{if the halo has just virialized.}$$

We can compute the redshift corresponding to t_{turn} using

$$t_{turn} = \frac{2}{3H_0}[1-(z+1)^{-3/2}] = t_0[1-(z+1)^{-3/2}]$$

which yields the following redshift

$$z = 2^{2/3} - 1 \approx 0.6$$

and a value of $z = 0.070$.

(e) The Universe was half as old as it is now, i.e. 6.5 Gyr old, when the Galaxy started to collapse.s

Problem 5: 2-point correlation function [15 points]

Question: *What would be the form of the galaxy 2-point correlation function if:*

a. Galaxies were distributed uniformly in space? [5 points]
b. All galaxies were on sheets/walls? [5 points]
c. All galaxies were in filaments? [5 points]

Solution :

The 2-point correlation function describes the probability of finding two galaxies separated by a distance r in some volume of space. This can be expressed as

$$dN(r) = n^2(1 + \xi(r))dV_1 dV_2,$$

where n is the number density of galaxies, dV_1, dV_2 are the respective volumes in which they can be found, $\xi(r)$ is the correlation function, and $dN(r)$ is the number of galaxies found. Here we're just interested in finding any two galaxies under the following conditions:

(a) If galaxies are distributed uniformly in three dimensions, then there is an equal chance to find a Galaxy at any distance from a given galaxy. Thus $\xi = 0$ in that case.

(b) If galaxies are distributed on sheets, then after the first galaxy is found, the second galaxy will have $n_2 \propto r^{-2}$ since it is confined to a 2-D sheet (V_2 is still proportinal to r^3). Thus the functional form of $\xi(r)$ will be

$$\xi(r) = \frac{B}{r} - 1$$

(c) If galaxies are distributed on filaments, that is, a 1-D string of galaxies, then $n_2 \propto r^{-1}$, and

$$\xi(r) = \frac{C}{r^2} - 1$$