Ay 21: Galaxies and Cosmology - Winter 2007 January 25, 2008

Solutions to Homework # 1

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These solutions are based on those of Thiago S. Gonçalves.

Problem 1: Apparent and absolute bolometric magnitudes [4 points each]

Question: *What are the apparent and absolute bolometric magnitudes of the following sources:*

Solution:

Let us first recall the definition of apparent magnitude :

$$m_1 - m_2 = -2.5 \log \frac{f_1}{f_2}$$

Where f_1 and f_2 are the fluxes received from the two sources.
The *absolute* magnitude of a source is the apparent magnitude it would have if placed at a distance of 10 pc from the observer. The apparent magnitude is related to the absolute magnitude in that way :

$$m = M - 5 + 5 \log(D/\text{pc})$$

Now it should be clear from those definitions that the absolute bolometric magnitudes of a source relates in that way the the Sun's absolute bolometric magnitude :

$$M - M_\odot = -2.5 \log \frac{L}{L_\odot}$$

(indeed when both sources are placed at the same distance of 10pc, the ratio of their received fluxes is just the ratio of their luminosities.)
The absolute bolometric magnitude of the Sun, $M_\odot = 4.7$, will serve as a reference along with its know luminosity $L_\odot = 3.8 \times 10^{33}$ erg.s^{-1}.

(a) A Sun-like star, 50 pc away :

The absolute magnitude is just $M_\odot = 4.7$. For $D = 50$ pc, we have :

$$m = M - 5 + 5 \log(D/\text{pc}) = 8.2$$

(b) A 100 Watt light bulb, 10 km away

First convert 100 $W = 100$ J.s$^{-1} = 10^9$erg.s^{-1}.

$$M_{LB} = M_\odot - 2.5 \log \frac{L_{LB}}{L_\odot} = 66.2$$

$$D = 10 \text{ km} = 3.2 \times 10^{-13} pc$$

$$m = 66.2 - 5 + 5 \log(3.2 \times 10^{-13}) = -1.3$$

(c) A galaxy containing $\sim 10^{11}$ **stars of an average luminosity** $0.5 L_\odot$ **, 20 Mpc away**

$$L_{Gal} = 5 \times 10^{10} L_\odot$$

$$M_{Gal} = M_\odot - 2.5 \log \frac{L_{Gal}}{L_\odot} = -22.0$$

$$D = 2 \times 10^7 pc$$

$$m = -22.0 - 5 + 5 \log(2 \times 10^7) = 9.5$$

(d) A quasar with luminosity $L_Q = 10^{46}$**erg.s**$^{-1}$**, 1 Gpc away**

$$M_Q = M_\odot - 2.5 \log \frac{10^{46}}{3.8 \times 10^{33}} = -26.3$$

$$m = -26.3 - 5 + 5 \log 10^9 = 13.7$$

By the way, (a) is a typical star in one of the major catalogs (HD, BD, SAO, etc.), (b) is like seeing house lights from a landing airplane, (c) is a bright galaxy in the Virgo cluster, (d) is similar to 3C273 or 3C48 (the first quasars discovered).

Problem 2a: The Olbers' Paradox [15 points]

Question: *Assume that you are in an infinitely old, infinitely large, static, Euclidean universe. The average density of stars is* $n_\star = 10^9$ *Mpc*$^{-3}$*, and the average stellar radius is* $R_\star = R_\odot = 7 \times 10^{10}$ *cm. How far on average would you have to look, before your line of sight would intercept a stellar surface? Now assume that the stars are clumped in galaxies with a mean density* $n_{Gal} = 1$ *Mpc*$^{-3}$*, and the average radius is* $R_{Gal} = 2$ *kpc. How far on average would you have to look, before your line of sight would intercept a galaxy?*

Solution:

We know that the probability of not finding an object in a given line of sight will decrease with increasing distance:

$$P = e^{-n\sigma l},$$

where n is the number density of the object in question and σ its cross-section. Thus, the average distance will be

$$< l >= \frac{\int_0^\infty l e^{-n\sigma l} dl}{\int_0^\infty e^{-n\sigma l} dl} = \frac{1}{n\sigma}.$$

We are given that, for stars, $R_\star = R_\odot = 7 \times 10^{10}$ cm (with the cross-section being $\sigma = \pi R^2$) and $n_\star = 1 \times 10^9$ Mpc^{-3}. Converting to the right units (1 Mpc = 3.0857×10^{24} cm), we find that

$$< l_\star >= \frac{1}{n_\star \pi R_\odot^2} = 6.185 \times 10^{17} \text{ Mpc} = 1.909 \times 10^{42} \text{ cm}.$$

For the second part of the problem, $n_{Gal} = 1$ Mpc^{-3} and $R_{Gal} = 2$ kpc, so

$$< l_{Gal} >= \frac{1}{n_{Gal} \pi R_{Gal}^2} = 79.58 \text{ Gpc} = 2.456 \times 10^{24} \text{ cm}.$$

Notice that, either way, the average distance is much larger than the radius of the universe (~ 4300 Mpc), and that helps to understand Olber's Paradox.

Problem 2b: The Olbers' Paradox [15 points]

Question: *In a uniformly filled 3-dimensional space, the number of stars encountered out to a radius* R *scales as* $N(R) \sim R^D$*, where* $D = 3$*; for a uniformly populated 2-dimensional space, that scaling would have* $D = 2$*; a fractal distribution embedded in a 3-dim. space would have* $D < 3$ *to be a non-integer number. For what (fractal) values of* D *would Olbers' Paradox disappear?*

Solution:

Assume a universe in which the number of stars seen at distance R is, as stated,

$$N(R) \propto R^D.$$

The increase in number at radius R is, then,

$$\frac{\mathrm{d}N(R)}{\mathrm{d}R} \propto R^{D-1}.$$

Multiplying by flux, which we knows scales as R^{-2}, yields

$$\frac{\mathrm{d}F(R)}{\mathrm{d}R} \propto R^{D-3}.$$

That is, stellar flux increases at a rate of R^{D-3}. We want that the total flux received in an infinite universe to be finite. In mathematical terms, that means

$$F = \int_0^\infty dF \propto \int_0^\infty R^{D-3} dR < \infty.$$

For the integral to converge, the exponent has to be less than -1, or

$$D < 2.$$

Problem 3: Derive the Friedmann Equation - Matter Only [30 points]

Question: *Derive (in a Newtonian approximation) the Friedmann equation for a universe with matter only (no radiation, pressure, or cosmological constant). Assume that the density is exactly equal to the critical value (aka the Einstein de Sitter model). What is the age vs. redshift relation in this universe? What is its present age in the units of* $t_H = \frac{1}{H_0}$*? What is the distance to* $z = 1$ *in units of* $D_H = \frac{c}{H_0}$*?*

Solution:

Imagine a homogeneous sphere of density ρ. The force of gravity on the surface of this sphere will yield an acceleration on a matter element so that

$$\ddot{r} = -\frac{GM}{r^2}.$$

We could multiply both sides by $\dot{r}$ and integrate to find that the sum of the kinetic and potential energies is constant:

$$\frac{1}{2}(\dot{r})^2 - \frac{GM}{r} = K.$$

Now, we know that for this sphere

$$M = \frac{4}{3}\pi r^3 \rho$$

which can be substituted into our last equation to yield

$$\frac{1}{2}(\dot{r})^2 = \frac{4}{3}\pi G \rho r^2 + K.$$

or

$$\left(\frac{\dot{r}}{r}\right)^2 = \frac{8\pi G}{3}\rho + \frac{K'}{r^2}.$$

For the case of critical density, the total energy is exactly zero, and the last term disappears. Rigorously speaking, that means that the curvature in such a universe is zero. The final answer can be expressed as:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho.$$

From the definition of redshift as ratio of scale factors,

$$1 + z \equiv \frac{1}{a}$$

and using the Friedmann equation in parametric form (meaning we are using Ω, the ratio of density to critical density, instead of ρ), it is easy to show that the lookback time is given by

$$t = \frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')E(z')},$$

where

$$E(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_{rad}(1+z)^4 + \Omega_\Lambda + \Omega_k(1+z)^2}.$$

In our case, only $\Omega_m = 1$ and all other terms cancel. Integrating, we get

$$t = \frac{2}{3H_0}\left[1 - (1+z)^{(-3/2)}\right].$$

To determine the age of the universe, simply remember that $a \rightarrow 0$ corresponds to $z \rightarrow \infty$. This means that

$$t_0 = \frac{2}{3H_0}.$$

To determine the comoving distance,

$$D_C = \int_{t_e}^{t_{obs}} \frac{c\,dt}{a(t)} = \frac{c}{H_0}\int_0^z \frac{dz'}{E(z')}.$$

$$D_C = \frac{2c}{H_0}\left[1 - (1+z)^{(-1/2)}\right]$$

$$\Rightarrow D_C(z=1) = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{2}}\right).$$

Problem 4: Derive the Friedmann Equation - Λ Only [20 points]

Question: *Now do the same, but for a universe with no matter, no radiation, and only the cosmological constant with the present density equal to the critical. Comment on the difference from the solutions to the previous problem.*

Solution:

This solution is no longer Newtonian, of course. However, it doesn't differ much from the previous case. If we replace matter density by energy density, the equation now looks like

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\epsilon}{3c^2}.$$

(Again, for a flat universe). Now, the major difference is that, since $\epsilon_w = \epsilon_{w,0}a^{-3(1+w)}$, and since $w = -1$ for the cosmological constant, then $\dot{a}/a$ is constant. The solution is an exponential of time.

To evaluate the lookback time, we have

$$t_{lookback} = \frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')} = \frac{1}{H_0}\log{(1+z)}.$$

Notice that, since $a(t) \propto e^t$, then a never goes to zero, which means no Big Bang. That is why

$$\lim_{z\to\infty} t_{lookback} = +\infty.$$

For comoving distance, we have

$$D_C = \frac{c}{H_0}\int_0^z dz' = \frac{cz}{H_0},$$

i.e. distances are much larger in this universe.