Ay 21: Galaxies and Cosmology - Winter 2008 February 27, 2008

Solutions to Homework # 4

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Problem 1: Collapse of a density fluctuation [30 points]

Question:
Consider a positive density fluctuation in the early universe, which is just bound, and thus it separates from the Hubble expansion, collapses and sheds excess kinetic energy, and virializes.
a. What is the change in its total energy? In its mean radius? Mean density? [12 poins]
b. Assuming that this happens in an Einstein - de Sitter universe, what is the final density contrast between the collapsed object and its surroundings? [18 points]

Solution:
(a.)
We compute the change in mean radius and mean density between the beginning of collapse (defined as the time when $\dot{R} = 0$ and the virialization.

The total energy is conserved in this process, as well as the total mass in the perturbation. So we have

$$E_f = E_i$$

Now initially,

$$E_i = -\frac{GM}{R_i}$$

(no kinetic energy). After the cluster virializes,

$$U + 2K = 0$$

which gives that

$$E_f = U + K = \frac{1}{2}U = -\frac{GM}{2R_f}$$

Thus, we get :

$$R_f = \frac{R_i}{2}$$

Consequently, the average density is given by :

$$\rho = \frac{M}{4\pi/3R^3}$$

so we get that

$$\rho_f = 8\rho_i$$

(b.)

The solution to this question is derived explicitly in lecture 8, pages 3 and 4. The final result is that the final density contrast between the collapsed object and its surrounding is :

$$\frac{\rho_{object}}{\rho_{background}} = 18\pi^2 \approx 180$$

Problem 2 : Peculiar velocities[14 points total, 7 each]

Question:

a. A cluster of galaxies is observed at a redshift cz = 5000 km/s. You are using a distance indicator whose zero-point which may vary by up to 20 % in different environments. What is the expected magnitude of a spurious peculiar velocity you may measure for this cluster?

b. You are surveying the volume out to cz = 3000 km/s. Assume that the typical peculiar velocities are of the order of 500 km/s. What precision of distance measurements do you need in order to have the peculiar velocities (not distances!) measured with a typical accuracy of 10% or better? How likely do you think this is, and why?

Solution :

(a.) The apparent total velocity of the cluster is given by :

$$cz \equiv V_{total} = V_{Hubble} + V_{pec} = H_0\ D + V_{pec}$$

Thus, if one measures cz and the distance to the cluster D, through a distance indicator, one can infer the peculiar velocity of this cluster :

$$V_{pec} = cz - H_0\ D$$

Now if the distance indicator has an error $\Delta D/D = 20\%$, this translates into an error in the measured peculiar velocity :

$$\Delta V_{pec} = H_0 \Delta D = H_0\ D\frac{\Delta D}{D} = (cz - V_{pec})\frac{\Delta D}{D} \approx cz\frac{\Delta D}{D}$$

In the last equality we assumed that V_{pec} is significantly smaller than cz. Plugging the numbers, $\Delta D/D = 0.2$ and $cz = 5000\ km/s$, we get :

$$\Delta V_{pec} \approx 1000\ km/s$$

(b.) We use the same formula as before :

$$\Delta V_{pec} \approx cz\frac{\Delta D}{D}$$

From this we infer that :

$$\frac{\Delta V_{pec}}{V_{pec}} \approx \frac{cz}{V_{pec}}\frac{\Delta D}{D}$$

We want $\Delta V_{pec}/V_{pec} < 10\%$ all over the survey. For this, we need

$$\frac{\Delta D}{D} \approx \frac{V_{pec}}{cz}\frac{\Delta V_{pec}}{V_{pec}} < 0.1 \times \frac{V_{pec}}{cz} < 0.1\frac{500\ km/s}{3000\ km/s} \approx 1.7\%$$

It is very unlikely to get such a precision : distance estimators are usually no more accurate than about 20% due to the variation in different environments.

Galaxy Cluster [20 points total, 5 points each]

Question:
Consider a cluster of galaxies with a radial velocity dispersion = 1500 km/s, and the mean radius R = 1.5 Mpc. It contains approx. 500 galaxies, with a mean luminosity $< L >= 10^{10} L_\odot$.

a. What is the estimated mass of the cluster?
b. What is the mass-to-light ratio, in Solar units?
c. What is the temperature of the intra-cluster gas (assume a pure hydrogen)?
d. What is the typical energy and wavelength of emitted photons (explain)?

Solution

(a) Using the virial theorem

$$2 < K > + < U >= 0$$

$$2\frac{1}{2}\sigma^2 = \frac{GM_{cluster}}{< R >}$$

we can get an order of magnitude estimate for the mass as

$$M_{cluster} \approx \frac{\sigma_r^2 < R >}{G}.$$

In reality, an astronomer is only able to observe radial velocities and there are likely to be velocities in either of the other two perpendicular directions, so $< v^2 >= 3\sigma_r^2$, and the potential energy will have to be adjusted based on the spatial distribution of galaxies in the cluster. But our order of magnitude estimate yields $M_{cluster} = 7.8 \times 10^{14} M_\odot$.

(b) The mass-to-light ration will be given by

$$\left(\frac{M_{cluster}}{L_{total}}\right) \approx \frac{M_{total}}{N < L >},$$

which gives $(M/L) \approx 157 M_{\odot}/L_{\odot}$.

(c) If we start with the equation for hydrostatic equilibrium

$$\frac{\partial P}{\partial r} = -\rho \frac{GM}{R},$$

we can integrate this to find

$$P = \frac{GM\rho}{R}.$$

Using the ideal gas law, we know

$$P = \frac{\rho k T}{\mu m_H},$$

which allows us to find T:

$$T = \frac{GM_{cluster} \mu m_H}{k < R >}.$$

If $\mu = 1$ for a pure$H$ gas, then $T = 2.7 \times 10^8 K$.

(d) If we assign an energy $E \approx kT$ to the photons emitted by the gas, then $E \approx 3.7 \times 10^{-8} ergs$. This corresponds to photons of wavelength $\sim 0.5\mathring{A}$, which falls in the X-ray region of the spectrum.

Problem 4 : Definitions[36 points total, 6 each]

Question *Define briefly (and be quantitative if you can) the following terms:*

a. Primordial density fluctuation spectrum
b. Free-fall time
c. Clusters of galaxies
d. Superclusters, voids, and filaments
e. Hierarchical clustering (or structure formation)
f. Peculiar velocity field

Solution

(a) One defines $\delta = (\rho + \delta\rho)/\rho$ as the density contrast between a perturbation and the background density. Then, after taking the Fourier transform δ_k, one can define the Power spectrum as $P(k) =< |\delta_k|^2 >$. The primordial power spectrum is the Power spectrum at early times, usually meaning after inflation.

(b) An object of mass M, initial radius R, will collapse under the influence of its own gravity, if there are no other balancing forces, in a time t_{ff} given by :

$$t_{ff} \approx \sqrt{\frac{R^3}{GM}}$$

(c) Galaxy clusters are the largest gravitationally bound objects. They contain 50 to 1000 Galaxies, have total masses of 10^{14} to $10^{15} M_{\odot}$. Some famous ones in this neck of the woods : the Virgo Cluster, Hercules Cluster, Coma Cluster...

(d) The Galaxies are not uniformly distributed in the Universe : they group in large, dense regions (clusters, superclusters), linked together by linear overdensities (filaments). The regions between those clusters and filaments are virtually empty, and called voids.

(e) Structure forms "bottom up" in the CDM model : small structures, as galaxies, form early, followed by clusters, which are still forming now, and will incorporate smaller structure as they collapse.

(f) Galaxies and clusters of galaxies are not in general fixed relative to the comoving coordinate restframe : they have, in addition to the velocity given by the Hubble flow, a random peculiar velocity.