Physics 106a – Final Exam – Due Dec 10, 2004

Instructions

Material: All lectures through Nov 22, all sections of lecture notes, Hand and Finch Chapters 1, 2, 3, 5, 6, 9, excluding sections not covered in lecture notes. Review the material ahead of time, consult myself, the TAs, your fellow students, or other texts if there is material you are having trouble with.

Logistics: Do not look at the exam until you are ready to start it. Please use a blue book if possible (makes grading easier), but there will be no penalty if you don't have one.

Time: 3 hrs at 100% credit + 3 hrs at 50% credit. Fixed time, you may take two 15 minute breaks at any point, but otherwise this should be a contiguous block of time. If you know the material well, you shouldn't need any of the extra 3 hrs. Clearly indicate the 100% credit and 50% credit parts of the exam. As a guide, I have indicated how long it took me to do each problem without having seen them before; total time for the entire exam was 125 minutes. Even if I see how to do the problem immediately, I can't write any faster than you, and I am also prone to lost factors of 2, sign flips, etc.

Reference policy: Hand and Finch, official class lecture notes, problem sets and solutions, your own lecture notes or other notes you have taken to help yourself understand the material. No other textbooks, no web searches, no interaction with your fellow students after you have looked at the exam. Calculators will not be needed. The algebra is much easier than on the midterm, symbolic manipulation programs will not be needed and are not allowed.

Question policy: Obviously, it will be difficult to ask questions if you are on a fixed-time policy. I have solved all the problems fully, so I have done my best to ensure there are no ambiguities. You are in the same boat as your fellow students, so if everyone has trouble with a particular problem, the grades will be curved accordingly.

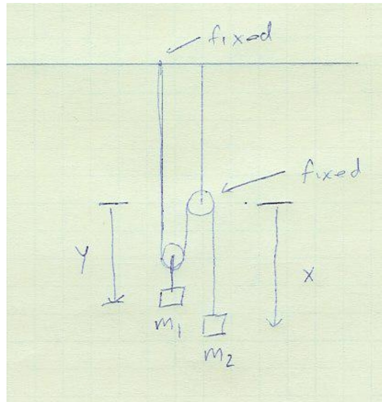
Due date: Friday, Dec 10, 5 pm, my office (311 Downs). 5 pm means 5 pm. **No late exams will be accepted!**

Grading weight: The exam is 40% of the class grade.

Suggestions on taking the exam:

- Go through and figure out roughly how to do each problem first; make sure you've got the physical concept straight before you start writing down formulae.
- Don't fixate on a particular problem. They are not all of equal difficulty. Come back to ones you are having difficulty with.
- Don't get buried in algebra. Get each problem to the point where you think you will get most of the points, then come back and worry about the algebra. Save the algebra for the 50% period.

1. (25 pts, 5 pts each part, 45 min. Part (e) takes almost half the time and is only worth 5 pts...) Consider the system in the following picture:



The rope is fixed at the left end. m_1 is rigidly attached to its pulley, which moves up and down with the rope. The other pulley is rigidly attached to the support. m_2 is attached to the rope and moves up and down with the rope. Gravity acts on both masses. The two pulleys should be assumed to be massless and of zero size. The rope is massless and of fixed length. **The problem should be considered one-dimensional** – that is, you can neglect any horizontal displacement of the pulleys or masses. This is **not** like Example 1.9 in the lecture notes.

(a) Find the Lagrangian for the system and obtain its Euler-Lagrange equation. Use the height of m_2 as one unconstrained coordinate. Reduce to an expression for $\ddot{x}$.

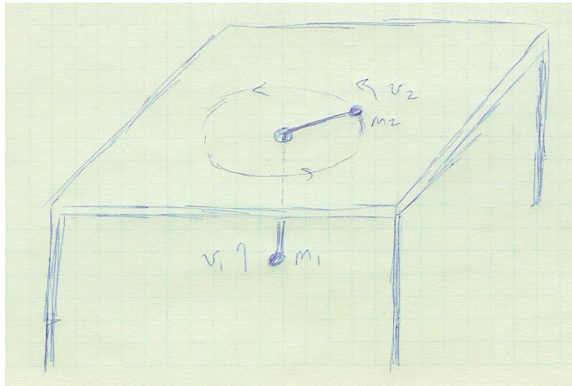
(b) Find the Hamiltonian for the system and write down Hamilton's equations. Make sure you use the correct independent variables! Show that the Hamilton's equations you obtain are equivalent to the Euler-Lagrange equation you found in (a) (of course, at this point you'll need to write Hamilton's equations in terms of x and $\dot{x}$ to show this).

(c) Find the acceleration of m_1 and m_2 using simple Newtonian mechanics. Find the tension in the rope in terms of m_1, m_2, and g *only*.

(d) Do the problem using Lagrange multipliers – *i.e.*, start with independent coordinates for the two masses and a constraint equation, obtain the Euler-Lagrange equations, find the accelerations of m_1 and m_2 and an expression for the Lagrange multiplier. Your three results should be written in terms of m_1, m_2 and g *only*. What is the physical significance of the Lagrange multiplier? Note that the sign and scaling of your Lagrange multiplier will depend on the sign and scale factor you choose for your constraint; don't worry if the Lagrange multiplier is off by a sign or scale factor from the "physically significant" quantity.

(e) Do the problem using the Hamilton-Jacobi formalism. That is, find relations between x, p_x (the canonical momentum conjugate to x) and t using the Hamilton-Jacobi formalism. Show that the acceleration in x is equivalent to that obtained by the other techniques. Note that you do not need to find an explicit solution or include initial conditions to demonstrate that $\ddot{x}$ is the same as found above. The integral you will find is trivially done, do not panic. (This is a test of whether you really understand Hamilton-Jacobi, one would never really use Hamilton-Jacobi to do a simple 1D problem.).

2. (30 pts, 5 pts each part, 25 min) Consider the system in the following picture:



Two point masses m_1 and m_2 are connected by a massless, inextensible rope of length L_0 through a tiny hole in a frictionless table. m_1 hangs below the table, suspended on the rope, acted upon by gravity. m_2 moves on the rope in an approximately circular path around the hole. Assume that m_1 is constrained to move in a vertical line, m_2 is constrainted to move only in the plane of the table, and the rope remains under tension.

(a) Write down the Lagrangian and find the Euler-Lagrange equations for the problem.

(b) If the initial conditions are $v_1 = 0$ and $v_2 = v_0$ (in circular motion), find the condition for stable circular orbits of m_2 about the hole.

(c) Write down the continuous symmetry transformation under which the system is invariant. Derive the conserved quantity. Hint: if you do this in the correct coordinate system (*i.e.*, don't just copy from the notes or H&F), it's really easy.

(d) Now, suppose the rope is slowly shortened in length from L_0 to L_1, but in such a way as to keep the height of m_1 fixed. (Neglect any oscillations from part (e) for the moment.) As the rope is shortened, are angular momentum and energy conserved? If either or both are not conserved, what causes it (them) to change? Calculate how v_2 depends on the initial speed v_0, the initial rope length L_0, the final rope length L_1, m_1, m_2, and g *only.*

(e) What is the frequency of small oscillations of the system about its equilibrium state?

(f) Next, suppose oscillations have been excited with amplitude δr and the rope is slowly shortened from L_0 to L_1 ("slowly" = the change per oscillation period is small compared to the oscillation amplitude), again holding the height of m_1 fixed. Calculate the change in the frequency and the amplitude of oscillation using adiabatic invariance and any conservation laws found in (c) and (d), again writing your solution *only* in terms v_0, L_0, L_1, m_1, m_2, and g.

3. (15 pts, 20 min) A mass m is hung from a fixed support by a spring of constant k. A second equal mass is hung from the first mass by an identical spring. Both springs are constrained to move vertically only. Gravity acts on both masses. Find the normal mode frequencies and normal mode vectors (up to normalization). Set up the equations to normalize the normal mode vectors and demonstrate they are orthogonal, but do not do the algebra. Do the normal mode frequencies or vectors depend on the gravitational acceleration g? Why or why not?

4. (5 pts, 5 min) Rewrite Hamilton's equations in terms of Poisson brackets, recalling that Poisson brackets can be used to obtain the time evolution of functions of the coordinates and momenta.

5. (10 pts, 10 min) Consider a simple pendulum with mass m and length l in the linear (small-angle) limit for the pendulum angle. Suppose l is a slowly varying function of time ($\dot{l}/l \ll$ pendulum frequency) and with $\dot{l}/l$ constant. Write down the Lagrangian and show that the equation of motion takes on the form of a damped simple harmonic oscillator. What is the Q of the oscillator? Calculate $\dot{\omega}/\omega$ in terms of $\dot{l}/l$. Calculate $\dot{E}/E$ in terms of $\dot{l}/l$ using adiabatic invariance, where E is the energy in the pendulum oscillation. Note that the $\dot{E}/E$ obtained is different than what one expects from the usual damped SHO relation, $\dot{E}/E = -1/Q$. How does this situation differ from the usual damped SHO problem (think about where the energy goes)?

6. (15 pts, 20 min) A taut string with tension τ and linear mass density λ is held fixed at $x = 0$ and is terminated at $x = L$ by a ring of negligible mass that slides without friction on a vertical rod.

   (a) (5 pts) Give a physical argument for why the boundary condition at $x = L$ is

   $$\left.\frac{\partial y}{\partial x}\right|_{x=L} = 0$$

   (b) (10 pts) Use the boundary conditions to obtain the allowed normal mode frequencies and the normal mode vectors; *i.e.*, find the equations analogous to

   $$\begin{aligned} \omega_n &= \frac{n\,\pi}{L}\sqrt{\frac{\tau}{\lambda}} \\ y_n(x,t) &= \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\,\pi\,x}{L}\right)\,\cos\left(\omega_n t\right) \end{aligned}$$

   Do not worry about the normalization – just get the spatial dependence of the modes.