Physics 106a – Problem Set 7 – Due Nov 30, 2004

Version 2

November 29, 2004

These problems cover the material on coupled harmonic oscillators and waves, from Hand and Finch Chapter 9 and Section 3.2 and 3.3 of the lecture notes (waves are covered **only** in the lecture notes, which are based on Thornton Chapter 13). Please again write down the rough amount of time you are spending on each problem.

Changes since v. 1: Corrected typos in the original formulae for problem 6. Changed $A(k)$ to $\alpha(k)$ and $\Psi(x,t)$ to $y(x,t)$ to match notation in lecture notes. An additional comment on "1/e width" added, and the definite integral of a Gaussian provided.

1. Hand and Finch 9.1. You may use the small-angle approximation.

2. Consider a thin homogeneous plate of mass M that lies in the xy plane with its center at the origin. Let the length of the plate be $2A$ (in the y direction) and let the width be $2B$ (in the x direction). The plate is suspended from a fixed support by four springs of equal force constant k at the four corners of the plate. The plate is free to oscillate but with the constraint that its center must remain on the z axis. Thus, we have three degrees of freedom: (1) vertical motion, with the center of the plate moving along the z axis; (2) a tipping motion lengthwise, with the x axis serving as an axis of rotation (choose an angle θ to describe this motion); and (3) a tipping motion sidewise, with the y axis serving as an axis of rotation (choose an angle ϕ to describe this motion). Assume only small oscillations and show that the normal mode frequency equation has a double root, and hence that the system is degenerate. Find the normal mode vectors of the system. (In evaluating the $\vec{\Phi}_k$ for the degenerate modes, arbitrarily set one component of one of the $\vec{\Phi}_k$ equal to zero to remove the indeterminacy.) Show that the degeneracy can be removed by adding to the plate a thin bar of mass m and length $2A$ situated (at equilibrium) along the y axis. Find the new normal mode frequencies (don't calculate the new mode vectors).

   Information you will need to calculate the kinetic and potential energies (and which you probably already know):

   - The moment of inertia of the plate for the rotations about the x and y axes are $I_x = \frac{1}{6}\,M\,A^2$ and $I_y = \frac{1}{6}\,M\,B^2$. Make sure you know which moment of inertia goes with which angle. The moment of inertia of the bar is $I_{bar} = \frac{1}{6}\,m\,A^2$.
   - The kinetic energy of a rotating object is $\frac{1}{2}\,I\,\Omega^2$ where Ω is the angular speed about the axis for which I is given.
   - Remember that potential energy of a spring-mass system is stored in the spring, not in the mass – using this fact will make it clear how to calculate the potential energy.

3. Obtain the normal mode frequencies for the double pendulum assuming equal lengths but not equal masses. Show that when the lower mass is small compared to the upper one, the

two resonant frequencies are almost equal. In this limiting case, if the pendula are set in motion by pulling the upper mass slightly away from the vertical and releasing it, show that subsequent motion is such that at regular intervals one pendulum is at rest while the other has its maximum amplitude. This is the familiar phenomenon of "beats."

4. A string is set into motion by being struck at a point $L/4$ from one end by a triangular hammer. The initial velocity is greatest at $x = L/4$ and decreases linearly to zero at $x = 0$ and $x = L/2$. The region $L/2 \leq x \leq L$ is initially undisturbed. Note that the string has zero displacement at $t = 0$ all along its length – the triangular hammer sets the velocity only. Determine the subsequent motion of the string. Why are the $n = 4, 8$ and related modes absent? By what factor down are the $n = 2, 3$ harmonics?

5. Consider an infinitely long continuous string with linear mass density λ_1 for $x < 0$ and for $x > L$, but density $\lambda_2 > \lambda_1$ for $0 < x < L$. If a wave train oscillating with an angular frequency ω is incident from the left on the high-density section of the string, find the reflected and transmitted intensities for the various portions of the string. Find a value of L that allows a maximum transmission through the high-density section. Discuss briefly the relationship of this problem to the application of nonreflective coatings to optical lenses. (For those of you who have taken quantum mechanics, notice the analogy to a particle incident on a potential barrier.)

6. Consider a wave packet with a Gaussian amplitude distribution

$$\alpha(k) = B \exp\left[-\sigma\,(k - k_0)^2\right]$$

where $2/\sqrt{\sigma}$ is equal to the $1/e$ width of the packet. (At the points $k = k_0 \pm 1/\sqrt{\sigma}$, the amplitude distribution is $1/e$ of its maximum value $\alpha(k_0)$. Thus $2/\sqrt{\sigma}$ is the width of the curve at the $1/e$ height.) Using this function for $\alpha(k)$, show that

$$\begin{aligned} y(x,0) &= B \int_{-\infty}^{\infty} dk \exp\left[-\sigma\,(k - k_0)^2\right] \exp\left(-ikx\right) \\ &= B\sqrt{\frac{\pi}{\sigma}} \exp\left[-\frac{x^2}{4\,\sigma}\right] \exp\left(-ik_0x\right) \end{aligned}$$

Sketch the shape of this wave packet. Next, expand $\omega(k)$ in a Taylor series, retain the first two terms, and integrate the wave packet equation to obtain the general result

$$y(x,t) = B\sqrt{\frac{\pi}{\sigma}} \exp\left\{-\frac{\left(\omega_0' t - x\right)^2}{4\,\sigma}\right\} \exp\left[i\left(\omega_0 t - k_0 x\right)\right]$$

Finally, take one additional term in the Taylor series expansion of $\omega(k)$ and show that σ is now replaced by a complex quantity. Find the expression for the $1/e$ width of the packet as a function of time for this case and show that the packet moves with the same group velocity as before but spreads in width as it moves. Illustrate this result with a sketch.

Note: in this problem, you will need the integral

$$\int_{-\infty}^{\infty} \exp\left(-a^2\,x^2\right) = \frac{\sqrt{\pi}}{a}$$