Physics 106a/196a – Problem Set 5 – Due Nov 11, 2005

Because of the midterm, we are covering two weeks of material in a single problem set. To keep the length of the problem set reasonable, some topics are not covered here:

- 106: Liouville's theorem
- 196: Lagrange multipliers for nonholonomic constraints and Legendre transformations. Symplectic notation and Poisson brackets are touched on only briefly in problem 6.

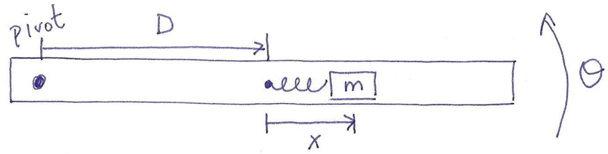
You are strongly encouraged to look in Hand and Finch and Thornton for problems to test your understanding of this material, as it will all be fair game for the final.
Problems 1 and 2 are for 106a students only, 3 and 4 for 106a and 196a students, and 5 and 6 for 196a students only.

1. (106) Consider the mass on the sliding inclined plane, Example 2.1.3 in Section 2.1.8 of the lecture notes. Repeat this problem using Lagrange multipliers to find the constraint force acting between the plane and the mass. Note that the natural coordinate along the constraint force direction is measured in the noninertial frame of the moving plane, so calculate the constraint force in both this frame and in the inertial frame of the floor. You may use the results in the lecture notes for this example to skip some algebraic steps, but make sure it is clear what you are doing.

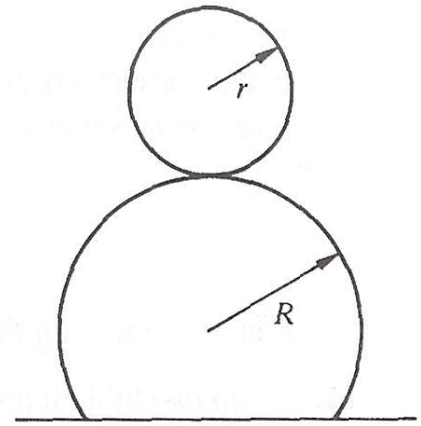
2. (106) A point mass m is attached to a spring of zero rest length and spring constant k. The spring is fastened at its end to a massless tube as shown, with the fastening point a distance D from a pivot point. The tube is free to rotate about the pivot point in the horizontal plane (see figure). There is no gravity in this problem, and there is no explicit driving of rotation in θ other than what is provided by the initial conditions. Find the Lagrangian, the canonical momenta, the Hamiltonian, and Hamilton's equations. Show that the equations of motion imply conservation of angular momentum and therefore that the equations of motion for x can be written

$$\dot{p}_x = -k\,x + \frac{p_\theta^2}{m(D+x)^3} \qquad \dot{x} = \frac{p_x}{m}$$

and so we have an effective 1-D equation of motion for x. Based on the above, write an equation that will give the equilibrium extension x_0 of the spring as a function of the initial angular momentum p_θ and the parameters D, k, and m (but do not solve). Taylor expand the equation of motion in x around this equilibrium point and find the frequency of small oscillations, leaving your expression in terms of x_0, p_θ, D, k, and m. Because angular momentum is conserved, there will be a relation between the amplitude of oscillations in x and oscillations in $\dot{\theta}$ (not θ itself). Find this relationship for small amplitude oscillations, again in terms of x_0, p_θ, D, k, and m.



Problem 2



Problem 3

3. (106/196) A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R as shown in the figure. The only external force is that of gravity. If the hoop starts rolling from rest on top of the bigger cylinder, use the method of Lagrange multipliers to find the point at which the hoop falls off the cylinder. Hint: See Thornton Example 7.10 for a simpler version of this problem. Your solution should reduce to the one for that problem in the limit $r \to 0$.

4. (106/196) Hand and Finch 5-11. When they ask "Is $H = T + V$?", one should interpret T to be $\frac{m\,c^2}{\sqrt{1-\frac{v^2}{c^2}}}$, which is actually the sum of the kinetic energy and the rest mass. A couple hints on part (b):

   - Remember that the Hamiltonian has to be written in terms of $\vec{p}$, not $\vec{v}$, and that there are v's in the many $\sqrt{1 - \frac{v^2}{c^2}}$ factors you will find.
   - You should obtain
   $$H = e\,\Phi + c\sqrt{\left(\vec{p} - \frac{e}{c}\,\vec{A}\right)^2 + m^2c^2}$$

   And one comment: while it is perfectly valid to write the Lagrangian and Hamiltonian in this way, it clearly treats time differently than spatial coordinates. Problem 5-6, which we did in class, shows how one can approach the Lagrangian in a way that treats time and space on equal footing.

5. (196) Hand and Finch 5-3. The thing that is tricky about this problem is reading enough information from the problem to write down the F_2-generating function partial differential equations: you are clearly told what $Z(z, P, t)$ is, so you can write down the first equation $Z = \frac{\partial F_2}{\partial P}$, but you are not explicitly given another relation that will let you write down the $p = \frac{\partial F_2}{\partial z}$ equation. (You know $p = m\,\dot{z}$ from the given form of the Hamiltonian, but you need $p(z, P, t)$: it is not allowed for $\dot{z}$ to appear in the transformation equations.) In such cases, you have to make a guess as to another relation between the z, Z, p, and P, solve for the generating function, and then check for contradictions – *e.g.*, make sure the generating function returns to you the relations you started with, make sure the Hamiltonian and resulting Hamilton's equations in Z and P are reasonable.

6. (196) By any method you choose, show that the following transformation is canonical:

$$x = \frac{1}{\alpha}\left(\sqrt{2\,P_1}\,\sin Q_1 + P_2\right) \qquad p_x = \frac{\alpha}{2}\left(\sqrt{2\,P_1}\,\cos Q_1 - Q_2\right)$$
$$y = \frac{1}{\alpha}\left(\sqrt{2\,P_1}\,\cos Q_1 + Q_2\right) \qquad p_y = -\frac{\alpha}{2}\left(\sqrt{2\,P_1}\,\sin Q_1 - P_2\right)$$

where α is some fixed parameter. (Yes, this results in a 4×4 Jacobian matrix, but enough elements are zero that testing for canonicity does not require major grunge.)

Apply this transformation to the problem of a particle of charge q moving in a plane that is perpendicular to a constant magnetic field $\vec{B}$ with

$$\alpha^2 = \frac{q\,B}{c}$$

Express the Hamiltonian for this problem in the $(\vec{Q}, \vec{P})$ coordinates and use Hamilton's equations to obtain the evolution of $(\vec{Q}, \vec{P})$ and thereby of (x, p_x, y, p_y). (Note that you don't need to find the generating function explicitly to find the new Hamiltonian – why?) What is the interpretation of each of the new variables $(\vec{Q}, \vec{P})$? Use the action-angle variable formalism to obtain a generating function for the transformation.