Lecture 2:
Linear Vector Spaces, Representations,
Linear Independence, Bases

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Prologue

We require a fair amount of mathematical machinery to discuss quantum mechanics:

- We must define the space that particle states live in.
- We must define what we mean by the operators that act on those states and give us physical observable quantities.
- We must explore the properties of these operators, primarily those properties that relate to Postulate 3, which says that an operator's eigenvalues are the only physically observable values for the associated physical variable.
- We must understand how states are normalized because of the important relation between the state vector and the relative probabilities of obtaining the spectrum of observable values for a given operator.
- We must also explore the operator analogues of symmetry transformations in classical mechanics; while these do not correspond to physical observables directly, we will see that they are *generated* by physical observables.

Prologue (cont.)

Why so much math? Well, in classical mechanics, we just deal with real numbers and functions of real numbers. You have been working with these objects for many years, have grown accustomed to them, and have good intuition for them. Being Caltech undergrads, calculus is a second language to you. So, in classical mechanics, you could largely rely on your existing mathematical base, with the addition of a few specific ideas like the calculus of variations, symmetry transformations, and tensors.

In QM, our postulates immediately introduce new mathematical concepts that, while having some relation to the 3D real vector space you are familiar with, are significant generalizations thereof. If we taught Hilbert spaces and operators from kindergarten, this would all be second nature to you. But we don't, so you now have to learn all of this math very quickly in order to begin to do QM.

Linear Vector Spaces: Definitions

Let us first discuss the idea of a linear vector space. A linear vector space $\mathbb{V}$ is a set of objects (called vectors, denoted by $|v\rangle$) and another associated set of objects called scalars (collectively known as a field), along with the following set of rules:

- The vectors have an addition operation (vector addition), $+$, that is closed, meaning that, for any $|v\rangle$ and $|w\rangle$ there exists a $|u\rangle$ in the vector space such that $|u\rangle = |v\rangle + |w\rangle$.

  We may also write the sum as $|v + w\rangle$.

  In defining the set of vectors that make up the vector space, one must also specify how addition works at an algorithmic level: when you add a particular $|v\rangle$ and $|w\rangle$, how do you know what $|u\rangle$ is?
- Vector addition is associative: $(|v\rangle + |w\rangle) + |u\rangle = |v\rangle + (|w\rangle + |u\rangle)$ for all $|u\rangle$, $|v\rangle$, and $|w\rangle$.
- Vector addition is commutative: $|v\rangle + |w\rangle = |w\rangle + |v\rangle$ for any $|v\rangle$ and $|w\rangle$.
- There is a unique vector additive zero or null or identity vector $|0\rangle$: $|v\rangle + |0\rangle = |v\rangle$ for any $|v\rangle$.
- Every vector has a unique vector additive inverse vector: for every $|v\rangle$ there exists a *unique* vector $-|v\rangle$ in the vector space such that $|v\rangle + (-|v\rangle) = |0\rangle$.

Linear Vector Spaces: Definitions (cont.)

- The scalars have an addition operation (scalar addition), $+$, that is closed, so that $a + b$ belongs to the scalar field if a and b do. The addition table must be specified.
- Scalar addition is associative: $a + (b + c) = (a + b) + c$ for any a, b, and c.
- Scalar addition is commutative: $a + b = b + a$ for any a, b.
- A unique scalar additive identity 0 exists: $a + 0 = a$ for any a.
- For any a, a unique scalar additive inverse $-a$ exists with $a + (-a) = 0$.
- The scalars have a multiplication operation (scalar multiplication) that is closed so that the product $a\,b$ belongs to the scalar field if a and b do. The multiplication table must be specified.
- Scalar multiplication is associative, $a\,(b\,c) = (a\,b)\,c$.
- Scalar multiplication is commutative, $a\,b = b\,a$.
- A unique scalar multiplication identity 1 exists: $1\,a = a$ for all a.
- For any $a \neq 0$, a unique scalar multiplicative inverse a^{-1} exists with $a\,a^{-1} = 1$.
- Scalar multiplication is distributive over scalar addition: $a\,(b + c) = a\,b + a\,c$.

Linear Vector Spaces: Definitions (cont.)

- There is a multiplication operation between vectors and scalars (scalar-vector multiplication) that is closed: For any vector $|v\rangle$ and any scalar α, the quantity $\alpha|v\rangle$ is a member of the vector space.

  We may also write the product as $|\alpha v\rangle$.

  Again, one must specify how this multiplication works at an algorithmic level.
- Scalar-vector multiplication is distributive in the obvious way over addition in the vector space: $\alpha\left(|v\rangle + |w\rangle\right) = \alpha|v\rangle + \alpha|w\rangle$
- Scalar-vector multiplication is distributive in the obvious way over addition in the field: $(\alpha + \beta)\,|v\rangle = \alpha|v\rangle + \beta|v\rangle$
- Scalar-vector multiplication is associative in the obvious way over multiplication in the field: $\alpha\left(\beta|v\rangle\right) = (\alpha\beta)\,|v\rangle$

Any vector of the form

$$|u\rangle = \alpha|v\rangle + \beta|w\rangle$$

is called a linear combination and belongs in the space according to the above rules.

Linear Vector Spaces: Definitions (cont.)

Shankar makes fewer assumptions than we do here and states that many of the properties of the scalar field we have assumed can in fact be derived. We choose to assume them because: a) the above assumptions are the standard mathematical definition of a field; and b) if one does not assume the above properties, one has to make some assumptions about how non-trivial the field arithmetic rules are in order to derive them. It's easier, and less prone to criticism by mathematicians, if we do as above rather than as Shankar.

Linear Vector Spaces: Definitions (cont.)

There are a few items that one needs to prove, though:

- The scalar addition identity 0 is consistent with the vector addition identity: $0|v\rangle = |0\rangle$

  Proof: $0|v\rangle + \alpha|v\rangle = (0+\alpha)\,|v\rangle = \alpha|v\rangle$. Since $|0\rangle + \alpha|v\rangle = \alpha|v\rangle$ already, and the identity element is unique, it holds that $0|v\rangle = |0\rangle$.

- Scalar-vector multiplication against the vector addition identity yields the obvious result $\alpha|0\rangle = |0\rangle$

  Proof: $\alpha|0\rangle + \alpha|v\rangle = \alpha\,(|0\rangle + |v\rangle) = \alpha|v\rangle$. Since $|0\rangle + \alpha|v\rangle = \alpha|v\rangle$ already, and the identity element is unique, it holds that $\alpha|0\rangle = |0\rangle$.

- The scalar multiplicative identity is the identity for scalar-vector multiplication also: $1|v\rangle = |v\rangle$

  Proof: $\alpha|v\rangle = (1\alpha)\,|v\rangle = 1\,(\alpha|v\rangle)$; α is arbitrary, so it holds for any $|v\rangle$ that $|v\rangle = 1|v\rangle$

- Vector additive inverses are consistent with scalar additive inverses: $(-1)\,|v\rangle = -|v\rangle$

  Proof: $(-1)\,|v\rangle + |v\rangle = (-1+1)\,|v\rangle = 0|v\rangle = |0\rangle$. Since inverses are unique, it holds that $(-1)\,|v\rangle = -|v\rangle$.

Example 3.1: Real vectors in N spatial dimensions, also known as $\mathbb{R}^N$

You are used to seeing real vectors in 3, and perhaps N, spatial dimensions, defined by the ordered triple

$$|v\rangle \leftrightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

where the $\{v_j\}$ are all real numbers (the reason for using $\leftrightarrow$ instead of $=$ will become clear in Example 3.4). The vector addition and scalar-vector multiplication algorithms are

$$|v\rangle + |w\rangle \leftrightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix}$$

$$\alpha|v\rangle \leftrightarrow \alpha \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \alpha\, v_1 \\ \alpha\, v_2 \\ \alpha\, v_3 \end{bmatrix}$$

Linear Vector Spaces: Examples (cont.)

Scalar addition and multiplication are just standard addition and multiplication of real numbers. All these operations are closed — *i.e.*, give back elements of the vector space — simply because addition and multiplication of real numbers is closed and because none of the operations change the "triplet" nature of the objects. Extension to N spatial dimensions is obvious. You should carry this example in your head as an intuitive representation of a linear vector space.

Example 3.2: Complex vectors in N spatial dimensions, also known as $\mathbb{C}^N$

Making our first stab at abstraction beyond your experience, let's consider complex vectors in N spatial dimensions. This consists of all ordered N-tuples

$$|v\rangle \leftrightarrow \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

where the $\{z_j\}$ are complex numbers, along with the same vector addition and scalar-vector multiplication rules as in the previous example. The space is closed by a logic similar to that used in the real vector space example.

This example is no more complicated than the real vector space $\mathbb{R}^N$. However, your intuition starts to break down because you will no doubt find it hard to visualize even the $N = 2$ example. You can try to imagine it to be something like real 2D space, but now you must allow multiplication by complex coefficients. The next obvious thing is to imagine it to be like real 4D space, but that's impossible to visualize. Moreover, it is misleading because it gives the impression that the space is 4-dimensional, but it really is only two-dimensional. Here is where you must start relying on the math and having only intuitive, not literal, pictures in your head.

Example 3.3: A spin-1/2 particle affixed to the origin

An interesting application of $\mathbb{C}^N$, and one that presents our first example of the somewhat confusing mathematics of quantum mechanics, is spin-1/2 particles such as the electron. As you no doubt learned in prior classes, such particles can be in a state of spin "up" along some spatial direction (say the z axis), spin "down" along that axis, or some linear combination of the two. (Later in the course, we will be rigorous by what we mean about that, but your intuition will suffice for now.) If we fix the particle at the origin so its only degree of freedom is the orientation of its spin axis, then the vector space of states of such particles consists of complex vectors with $N = 2$:

$$|\psi\rangle \leftrightarrow \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

A particle is in a pure spin-up state if $z_2 = 0$ and in a pure spin-down state if $z_1 = 0$.

Linear Vector Spaces: Examples (cont.)

There are many weirdnesses here:

- The particle state can be a linear combination of these two states:

$$|\psi\rangle \leftrightarrow z_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + z_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

  The state is neither perfectly= spin up or perfectly spin down. We will frequently write this state as

$$|\psi\rangle = z_1 \langle\uparrow| + z_2 \langle\downarrow| \quad \text{with} \quad \langle\uparrow| \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \langle\downarrow| \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- The particle *lives* in $\mathbb{R}^3$, a real vector space of $N = 3$ dimensions, in that we measure the orientation of its spin axis relative to the axes of that vector space. But the vector space of its quantum mechanical states is $\mathbb{C}^2$, the complex vector space of $N = 2$ dimensions. *The space of QM states is distinct from the space the particle "lives" in!*

Example 3.4: The set of all complex-valued functions on a set of discrete points $i\,\frac{L}{n+1}$, $i = 1, \ldots, n$, in the interval $(0, L)$

You are well aware of the idea of a complex function $f(x)$ on an interval $(0, L)$. Here, let's consider a simpler thing, a function an a set of equally spaced, discrete points. The vector $|f\rangle$ corresponding to a particular function is then just

$$|f\rangle \leftrightarrow \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} \quad \text{with} \quad x_j = j\,\frac{L}{N+1}$$

You are used to taking linear combinations of functions,

$$h(x) = \alpha\, f(x) + \beta\, g(x)$$

We can do the same thing with these vector objects:

$$|h\rangle = \alpha\,|f\rangle + \beta\,|g\rangle \leftrightarrow \alpha \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} + \beta \begin{bmatrix} g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix} = \begin{bmatrix} \alpha\, f(x_1) + \beta\, g(x_1) \\ \vdots \\ \alpha\, f(x_n) + \beta\, g(x_N) \end{bmatrix}$$

Linear Vector Spaces: Examples (cont.)

It is hopefully obvious that this is just a more complicated way of writing $\mathbb{C}^N$: the space is just the set of N-tuples of complex numbers, and, since we can define any function of the $\{x_j\}$ that we want, we can obtain any member of $\mathbb{C}^N$ that we want.

This example lets us introduce the concept of a representation. Given a set of objects and a set of rules for their arithmetic — such as the vector space $\mathbb{C}^N$ — a representation is a way of writing the objects down on paper and expressing the rules. One way of writing down $\mathbb{C}^N$ is simply as the set of all N-tuples of complex numbers. Another way is as the set of all linear combinations of x^a for $a = 1, \ldots, N$ on these discrete points. To give a specific example in $\mathbb{C}^3$:

$$\begin{bmatrix} (1/4)\,L \\ (1/2)\,L \\ (3/4)\,L \end{bmatrix} \leftrightarrow |u\rangle \leftrightarrow x \qquad \begin{bmatrix} (1/16)\,L \\ (1/4)\,L \\ (9/16)\,L \end{bmatrix} \leftrightarrow |v\rangle \leftrightarrow x^2 \qquad \begin{bmatrix} (1/64)\,L \\ (1/8)\,L \\ (27/64)\,L \end{bmatrix} \leftrightarrow |w\rangle \leftrightarrow x^3$$

The vector space elements are $|u\rangle$, $|v\rangle$, and $|w\rangle$. In the column-matrix representation, they are represented by the column matrices. In the functional representation, they are represented by the given functions. We use $\leftrightarrow$ to indicate "represented by" to distinguish it from "equality".

The alert reader will note that the representation in terms of functions is not one-to-one — it is easy to make two functions match up at 3 points but be different elsewhere. We will not worry about this issue now, it will matter later.

Linear Vector Spaces: Examples (cont.)

An aspect of the concept of representation that is confusing is that we usually need to write down a representation to initially define a space. Here, to define $\mathbb{C}^N$, we needed to provide the representation in terms of complex N-tuples. But the space $\mathbb{C}^N$ is more general than this representation, as indicated by the fact that one can write $\mathbb{C}^N$ in terms of the function representation. The space takes on an existence beyond the representation by which it was defined.

In addition to introducing the concept of a representation, this example will become useful as a lead-in to quantum mechanics. You can think of these vectors as the QM wavefunction (something we will define later) for a particle that lives only on these discrete sites $\{x_j\}$. We will eventually take the limit as the spacing $\Delta = \frac{L}{N+1}$ vanishes and N becomes infinite, leaving the length of the interval fixed at L but letting the function now take on a value at any position in the interval $[0, L]$. This will provide the wavefunction for a particle confined in a box of length L.

Finally, one must again be careful not to confuse the space that the particle lives in with the space of its quantum mechanical states. In this case, the former is set of n points on a 1-dimensional line in $\mathbb{R}^1$, while the latter is $\mathbb{C}^N$. When we take the limit $\Delta \to 0$, the particle will then live in the interval $[0, L]$ in $\mathbb{R}^1$, but its space of states will become infinite-dimensional!

Example 3.5: The set of real, antisymmetric $N \times N$ square matrices with the real numbers as the field.

Antisymmetric matrices satisfy $A^T = -A$ where ^{T} indicates matrix transposition. For $N = 3$, these matrices are of the form

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$$

The vector addition operation is standard matrix addition, element-by-element addition. The scalar arithmetic rules are just addition and multiplication on the real numbers. The scalar-multiplication operation is multiplication of all elements of the matrix by the scalar.

Linear Vector Spaces: Examples (cont.)

It is easy to see that this set satisfies all the vector space rules:

- The sum of two antisymmetric matrices is clearly antisymmetric.
- Addition of matrices is commutative and associative because the element-by-element addition operation is.
- The null vector is the matrix with all zeros.
- The additive inverse is obtained by taking the additive inverse of each element.
- Multiplication of a real, antisymmetric matrix by a real number yields a real, antisymmetric matrix.
- Scalar-vector multiplication is distributive because the a scalar multiplies every element of the matrix one-by-one.
- Scalar-vector multiplication is associative for the same reason.

Note that standard matrix multiplication is *not* included as one of the arithmetic operations here! You can check that the space is not closed under that operation.

Linear Vector Spaces: Examples (cont.)

This example provides a more subtle version of the concept of a representation. There are two aspects to discuss here. First, the example shows that one need not write a vector space in terms of simple column matrices. Here, we use $N \times N$ square matrices instead. The key is whether the objects satisfy the linear vector space rules, not the form in which the objects are written. Second, one can see that this vector space is a representation of $\mathbb{R}^{N(N-1)/2}$: any element has $N(N-1)/2$ real numbers that define it, and the arithmetic rules for matrix addition and scalar multiplication and addition are consistent with the corresponding rules for column-matrix addition and scalar multiplication and addition.

Clearly, one must learn to generalize, to think abstractly beyond a representation of these mathematical objects to the objects themselves. The representation is just what you write down to do calculations, but the rules for the objects are more generic than the representation.

Linear Vector Spaces: Linear Independence and Bases

What is the minimal set of vectors needed to construct all the remaining vectors in a vector space? This question brings us to the concepts of linear independence and of a basis for the vector space.

A set of vectors $\{|v_j\rangle\}$ is linearly independent if no one of them can be written in terms of the others. Mathematically: there is no solution to the equation

$$\sum_{j=1}^{n} \alpha_j |v_j\rangle = 0 \tag{3.1}$$

except $\alpha_j = 0$ for all j. The rationale for this definition is straightforward: suppose there were such a set of $\{\alpha_j\}$, and suppose without loss of generality that $\alpha_1 \neq 0$. Then we can rewrite the above as

$$|v_1\rangle = \frac{1}{\alpha_1} \sum_{j=2}^{n} \alpha_j |v_j\rangle \tag{3.2}$$

thereby rewriting $|v_1\rangle$ in terms of the others.

A vector space is defined to have dimension n if the maximal set of linearly independent vectors that can be found has n members.

Linear Vector Spaces: Linear Independence and Bases (cont.)

We next state two important expansion theorems (The proofs are straightforward, you can look them up in Shankar).

- Given a set of n linearly independent vectors $\{|v_j\rangle\}$ in a n-dimensional vector space, any other vector $|v\rangle$ in the vector space can be expanded in terms of them:

$$|v\rangle = \sum_j \alpha_j |v_j\rangle \tag{3.3}$$

- The above expansion is unique.

Because of the above expansion theorems, any such set of n linearly independent vectors is called a basis for the vector space and is said to span the vector space. The coefficients $\{\alpha_j\}$ for a particular vector $|v\rangle$ are called the components of $|v\rangle$. Equation 3.3 is termed the (linear) expansion of $|v\rangle$ in terms of the basis $\{|v_j\rangle\}$. The vector space is said to be the space spanned by the basis.

Note that, by definition, the concept of linear independence and the linear expansion are representation-independent — both concepts are defined in terms of the vectors and the field elements, not in terms of representations. As usual, you must usually pick a representation to explicitly test for linear independence or to calculate expansion coefficients, but the result must be representation-independent because the definitions are.

Example 3.6: The real and complex vectors on N spatial dimensions

The obvious basis for both of these spaces is

$$|1\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad |2\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad |N\rangle \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Other bases are possible, though. For example

$$|1'\rangle \leftrightarrow \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad |2'\rangle \leftrightarrow \begin{bmatrix} 1 \\ -1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad |(N-1)'\rangle \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \qquad |N'\rangle \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{bmatrix}$$

Linear Vector Spaces: Linear Independence and Bases (cont.)

One can prove linear independence by writing down Equation 3.1, giving N equations in the N unknowns $\{\alpha_j\}$ and solving. The first basis just yields the N equations $\alpha_j = 0$ for each j, which implies linear independence. Try the second basis for yourself.

In addition, one can show that $\mathbb{R}^N$ and $\mathbb{C}^N$ are N-dimensional by trying to create a $(N+1)$-dimensional basis. We add to the set an arbitrary vector

$$|v\rangle \leftrightarrow \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}$$

where the $\{v_j\}$ are real for $\mathbb{R}^N$ and complex for $\mathbb{C}^N$, and set up Equation 3.1 again. If we use the first basis $\{|j\rangle\}$, one obtains the solution $\alpha_j = v_j$, indicating that any $|v\rangle$ is not linearly independent of the existing set. Since there are N elements of the existing set, the space is N-dimensional.

Note that this proves that $\mathbb{C}^N$, as defined, with a complex field, is N-dimensional, not $2N$-dimensional. If one restricts the field for $\mathbb{C}^N$ to real numbers, then one requires a set of N purely real basis elements and N purely imaginary basis elements, yielding a $2N$-dimensional space. But that is a different space than the one we defined; with a complex field, $\mathbb{C}^N$ is without a doubt N-dimensional.

Example 3.7: Spin-1/2 particle at the origin

We saw in Example 2 that this space is just $\mathbb{C}^2$. Here, though, it is useful to get into the physics of different bases. We already stated (without explanation) that the usual orthonormal basis for this space corresponds to spin up and spin down relative to the physical z axis:

$$|\uparrow_z\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |\downarrow_z\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Two other reasonable bases are

$$|\uparrow_x\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad |\downarrow_x\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|\uparrow_y\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \qquad |\downarrow_y\rangle \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

where $i = \sqrt{-1}$ here.

As the notation suggests, $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$ correspond, respectively, to a particle in a spin up or spin down state relative to the physical x axis, and, similarly, $|\uparrow_y\rangle$ and $|\downarrow_y\rangle$ are the same for the physical y axis. We shall see how these different bases arise as *eigenvectors* of, respectively, the z, x, and y axis spin operators S_z, S_x, and S_y. One can immediately see that, if a particle is in a state of definite spin relative to one axis, it cannot be in a state of definite spin with respect to another — *e.g.*, $|\uparrow_x\rangle = (|\uparrow_z\rangle + |\downarrow_z\rangle)/\sqrt{2}$. This inability to specify spin along multiple axes simultaneously reflects the fact that the corresponding spin operators do not commute, a defining property of quantum mechanics. Much more on this later; certainly, rest assured that this mathematical discussion has significant physical implications.

Following the linear expansion formulae, we can expand the elements of any basis in terms of any other basis; *e.g.*:

$$|\uparrow_y\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow_z\rangle + i\,|\downarrow_z\rangle\right] \qquad |\downarrow_y\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow_z\rangle - i\,|\downarrow_z\rangle\right]$$

$$|\uparrow_z\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow_y\rangle + |\downarrow_y\rangle\right] \qquad |\downarrow_z\rangle = \frac{-i}{\sqrt{2}}\left[|\uparrow_y\rangle - |\downarrow_y\rangle\right]$$

Example 3.8: The set of all complex-valued functions on a set of discrete points $i\,\frac{L}{n+1}$, $i = 1, \ldots, n$, in the interval $(0, L)$, as in Example 3.4

As we have discussed, this space is the same as $\mathbb{C}^N$. The first basis given in the previous example for $\mathbb{C}^N$ is fine and has the advantage of being physically interpreted as having the particle localized at one of the N discrete points: $|i\,\rangle$ corresponds to the particle being at $x_j = j\,L/(N+1)$. But another basis is the one corresponding to the power law functions x^a, $a = 1, \ldots, N$. For $N = 3$, the representations are

$$x \leftrightarrow |1\,\rangle \leftrightarrow \begin{bmatrix} (1/4)\,L \\ (1/2)\,L \\ (3/4)\,L \end{bmatrix} \quad x^2 \leftrightarrow |2\,\rangle \leftrightarrow \begin{bmatrix} (1/16)\,L \\ (1/4)\,L \\ (9/16)\,L \end{bmatrix} \quad x^3 \leftrightarrow |3\,\rangle \leftrightarrow \begin{bmatrix} (1/64)\,L \\ (1/8)\,L \\ (27/64)\,L \end{bmatrix}$$

If one writes down Equation 3.1, one can show that the only solution is, again, $\alpha_j = 0$ for all i. Let's write the three equations as a matrix equation:

$$\begin{bmatrix} 1/4 & 1/16 & 1/64 \\ 1/2 & 1/4 & 1/8 \\ 3/4 & 9/16 & 27/64 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Recall your linear algebra here: the solution for the $\{\alpha_j\}$ is only nontrivial if the determinant of the matrix vanishes. It does not, so $\alpha_j = 0$ for all i.

Example 3.9: The set of real, antisymmetric $N \times N$ matrices (with the real numbers as the field) as in Example 3.5.

This space is a vector space of dimension $N(N-1)/2$ with one possible basis set just being the real, antisymmetric matrices with two nonzero elements each; for example, for $N = 3$, we have

$$|1\rangle \leftrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad |2\rangle \leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad |3\rangle \leftrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

As with $\mathbb{R}^N$ and $\mathbb{C}^N$, there are many other possible bases. One alternative is

$$|1'\rangle \leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad |2'\rangle \leftrightarrow \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad |3'\rangle \leftrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

One can check that both sets are linearly independent.