Lecture 8:
The Eigenvector-Eigenvalue Problem Continued

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Degeneracy

What happens when two or more eigenvalues are equal? Intuitively, one sees that, if there were two eigenvectors $|\omega, 1\rangle$ and $|\omega, 2\rangle$ corresponding to the same eigenvalue $\omega$, then any linear combination would also be an eigenvector with the same eigenvalue:

$$\text{if } A|\omega, 1\rangle = \omega|\omega, 1\rangle \quad \text{and} \quad A|\omega, 2\rangle = \omega|\omega, 2\rangle \quad (3.64)$$

$$\text{then } A(\alpha|\omega, 1\rangle + \beta|\omega, 2\rangle) = \alpha \omega|\omega, 1\rangle + \beta \omega|\omega, 2\rangle = \omega (\alpha|\omega, 1\rangle + \beta|\omega, 2\rangle)$$

Hence, one expects that the formalism should be unable to pick between $|\omega, 1\rangle$, $|\omega, 2\rangle$, and any linear combination of the two. It in fact does have problems; in general, rather than there being just one redundant equation when one solves for the eigenvector, there are $n_d$ redundant equations where $n_d$ is the number of degenerate eigenvalues. This is to be expected, as what the problem is saying is that all vectors in a subspace of dimension $n_d$ are eigenvectors, and it’s therefore entirely arbitrary which $n_d$ of those vectors one chooses to be the nominal eigenvectors. Of course, if one wants to span the subspace, one had better pick linearly independent ones.

We will show below that any pair of eigenvectors corresponding to nondegenerate eigenvalues are always orthogonal. Motivated by this, the usual procedure is to pick a convenient set of orthogonal vectors in the degenerate subspace as the eigenvectors. They are automatically orthogonal to the other, nondegenerate eigenvectors, and making them orthogonal provides an overall orthogonal (and hence easily orthonormalizable) basis for the inner product space.
Theorems on Properties of Eigenvalues and Eigenvectors

The eigenvalues of a Hermitian operator are real.

Assume the Hermitian operator $\Omega$ has eigenvalue $\omega$ with eigenvector $|\omega\rangle$, $\Omega|\omega\rangle = \omega|\omega\rangle$. Take the matrix element of $\Omega$ between the ket $|\omega\rangle$ and bra $\langle\omega|$ (also known as the expectation value of $\Omega$ as we shall see later):

$$\langle\omega|\Omega|\omega\rangle = \omega\langle\omega|\omega\rangle \quad (3.65)$$

Also consider the adjoint of the above expression

$$\langle\omega|\Omega^\dagger|\omega\rangle = \omega^*\langle\omega|\omega\rangle \quad (3.66)$$

The two expressions must be equal because $\Omega^\dagger = \Omega$, so we have

$$(\omega - \omega^*)\langle\omega|\omega\rangle = 0 \quad (3.67)$$

Unless $\langle\omega|\omega\rangle = 0$, which can only hold for $|\omega\rangle = |0\rangle$, implying a trivial operator $\Omega$, we find that $\omega = \omega^*$; i.e., the eigenvalue $\omega$ is real.
Any pair of eigenvectors corresponding to nondegenerate eigenvalues of a Hermitian operator are orthogonal.

Given two eigenvalues $\omega_j$ and $\omega_k$ and corresponding eigenvectors $|\omega_j\rangle$ and $|\omega_k\rangle$, we have

$$\langle \omega_j | \Omega | \omega_k \rangle = \langle \omega_j | \omega_k | \omega_k \rangle = \omega_k \langle \omega_j | \omega_k \rangle$$  \hspace{1cm} (3.68)

and

$$\langle \omega_j | \Omega | \omega_k \rangle = \left( \Omega^\dagger |\omega_j\rangle \right)^\dagger |\omega_k\rangle = (\Omega |\omega_j\rangle)^\dagger |\omega_k\rangle = (\omega_j |\omega_j\rangle)^\dagger |\omega_k\rangle = \langle \omega_j | \omega_j^* | \omega_k \rangle$$

$$= \langle \omega_j | \omega_j | \omega_k \rangle = \omega_j \langle \omega_j | \omega_k \rangle$$  \hspace{1cm} (3.69)

where we have used that $\Omega$ is Hermitian and that its eigenvalue $\omega_j$ is real. We thus have

$$(\omega_j - \omega_k) \langle \omega_j | \omega_k \rangle = 0$$  \hspace{1cm} (3.70)

Because we assumed nondegenerate eigenvalues $\omega_j \neq \omega_k$, we have $\langle \omega_j | \omega_k \rangle = 0$. 

For any Hermitian operator acting on an inner product space with a complex field, there exists an orthonormal basis of its eigenvectors, termed its eigenbasis.

We will first prove this for the case of no degenerate eigenvalues. Our proof is somewhat different than Shankar’s.

The proof is almost trivial. Any Hermitian operator acting on a $n$-dimensional inner product space with a complex field has $n$ eigenvalues because the operator has a $n \times n$ matrix representation, yielding a characteristic polynomial of $n$th order. As mentioned before, it is guaranteed to have $n$ complex roots. There are thus $n$ eigenvalues, nondegenerate by assumption here.

We have shown that, for nondegenerate eigenvalues, the eigenvectors of any pair of eigenvalues are orthogonal. We are thus assured of a mutually orthogonal set of $n$ eigenvectors. It is trivial to render these orthonormal by picking their normalization appropriately (the length of an eigenvector is arbitrary, recall).

Finally, because our orthonormal set is clearly linearly independent, and because it contains $n$ vectors, it is a valid basis for the $n$-dimensional inner product space.
When represented in terms of a basis of its eigenvectors, a Hermitian operator’s matrix representation is diagonal and its diagonal elements are its eigenvalues.

Again, we take a different tack than Shankar. If we write $\Omega$ in a matrix representation in which its eigenvectors are the basis, then its eigenvectors have matrix representation

$$\begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix} \leftrightarrow |\omega_1\rangle \quad \begin{pmatrix}
0 \\
1 \\
\vdots \\
0
\end{pmatrix} \leftrightarrow |\omega_2\rangle \quad \cdots \quad \begin{pmatrix}
0 \\
0 \\
\vdots \\
1
\end{pmatrix} \leftrightarrow |\omega_n\rangle$$

The matrix representation of an operator in a particular basis’s matrix representation is given by the matrix elements of the operator between the basis members according to Equation \ref{eq:3.37}. So, here we have

$$\Omega \leftrightarrow \begin{bmatrix}
\langle \omega_1 | \Omega | \omega_1 \rangle & \cdots & \langle \omega_1 | \Omega | \omega_n \rangle \\
\vdots & \ddots & \vdots \\
\langle \omega_n | \Omega | \omega_1 \rangle & \cdots & \langle \omega_n | \Omega | \omega_n \rangle
\end{bmatrix} = \begin{bmatrix}
\omega_1 & 0 & \cdots & 0 \\
0 & \omega_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \omega_n
\end{bmatrix} \quad (3.71)$$

because the basis elements are eigenvectors of $\Omega$ and form an orthonormal set; that is, because $\Omega_{jk} = \langle \omega_j | \Omega | \omega_k \rangle = \omega_k \langle \omega_j | \omega_k \rangle = \omega_k \delta_{jk}$.
Recalling our bilinear form for operators, Equation 3.42, we may also use the condition $\Omega_{j,k} = \omega_j \delta_{j,k}$ on the matrix elements of $\Omega$ in its eigenbasis to write the operator in the form

$$\Omega = \sum_{j,k=1}^{n} |\omega_j \rangle \langle \omega_j | \Omega |\omega_k \rangle \langle \omega_k | = \sum_{j,k=1}^{n} |\omega_j \rangle \omega_j \delta_{j,k} \langle \omega_k | = \sum_{j=1}^{n} \omega_j |\omega_j \rangle \langle \omega_j |$$

(3.72)

This makes it explicit that $\Omega$'s matrix representation is diagonal when the basis for the matrix representation is $\Omega$'s eigenbasis.
Degenerate case:

Even if one has degenerate eigenvalues, the above results still hold – one can still construct an orthonormal basis of the operator’s eigenvectors, and then one can write the matrix representation of the operator and it is diagonal. We are not going to be strictly rigorous about proving this, but we can make a fairly ironclad argument.

Let \( \omega \) be an eigenvalue that is \( n_d \) times degenerate. We know that the set of vectors that are eigenvectors with this eigenvalue form a subspace because the set is closed under linear combinations, as we noted earlier (the other arithmetic properties of the subspace are inherited from the parent space.)

Let us assume for the moment that \( \omega \) is the only degenerate eigenvalue, so that there are \( n_n = n - n_d \) nondegenerate eigenvalues. This provides \( n_n \) mutually orthogonal eigenvectors as shown above. Note also that our eigenvector orthogonality proof also implies that these nondegenerate eigenvectors are orthogonal to any vector in the \( \omega \) subspace because any vector in that subspace is an eigenvector of \( \Omega \) with eigenvalue \( \omega \), which is a different eigenvalue from any of the \( n_n \) nondegenerate eigenvalues, and hence the previously given proof of orthogonality carries through.

We thus have a \( n \)-dimensional vector space with a subspace of dimension \( n_n = n - n_d \). We make the intuitively obvious claim that the remaining subspace, which is the degenerate subspace, thus has dimension \( n_d \) and therefore has at least one linearly independent basis set with \( n_d \) elements.
Finally, we invoke Gram-Schmidt orthogonalization to turn that linearly independent basis into an orthonormal basis. This basis for the degenerate subspace is automatically orthogonal to the eigenvectors with nondegenerate eigenvalues, so together they form an orthonormal basis for the entire space.

If there is more than one degenerate eigenvalue, one simply performs the above procedure for each degenerate subspace independently.
The eigenvectors of a unitary operator are complex numbers of unit modulus.

Consider the norm of an eigenvector $|\omega\rangle$ of the unitary operator with eigenvalue $\omega$:

$$\langle \omega | \omega \rangle = \langle \omega | U^\dagger U | \omega \rangle = \langle \omega | \omega^* \omega | \omega \rangle \implies (\omega^* \omega - 1) \langle \omega | \omega \rangle = 0 \quad (3.73)$$

For nontrivial $|\omega\rangle$, we have $\omega^* \omega = 1$, and hence $\omega$ must have unit modulus. (This last step you can prove by writing $\omega$ out in terms of real and imaginary components and solving the equation for its components.)

The eigenvectors of a unitary operator are mutually orthogonal.

Consider a similar construct, this time the inner product of eigenvectors $|\omega_j\rangle$ and $|\omega_k\rangle$ of two nondegenerate eigenvalues $\omega_j \neq \omega_k$:

$$\langle \omega_j | \omega_k \rangle = \langle \omega_j | U^\dagger U | \omega_k \rangle = \langle \omega | \omega_j^* \omega_k | \omega \rangle \implies (\omega_j^* \omega_k - 1) \langle \omega_j | \omega_k \rangle = 0 \quad (3.74)$$

For $\omega_j \neq \omega_k$, the quantity $\omega_j^* \omega_k - 1$ cannot vanish unless $\omega_j = \omega_k$, which we assumed did not hold. Therefore $\langle \omega_j | \omega_k \rangle = 0$ and we have orthogonality.

Of course, we can deal with degenerate subspaces in the same way as we did for Hermitian operators.
Diagonalization of Hermitian Matrices and Unitary Transformations

Since we have shown that one can always construct an orthonormal basis of the eigenvectors of a Hermitian matrix, we can write down a unitary operator whose matrix representation in the original basis \( \{|j\rangle\} \) is made up from the components of those eigenvectors in that basis:

\[
U_\Omega \leftrightarrow \begin{bmatrix}
\langle 1 | \omega_1 \rangle & \cdots & \langle 1 | \omega_n \rangle \\
\vdots & \ddots & \vdots \\
\langle n | \omega_1 \rangle & \cdots & \langle n | \omega_n \rangle
\end{bmatrix}
\]  

(3.75)

That is, we make up each column of the matrix from the expansion coefficients of the eigenvectors in the original basis \( \{|j\rangle\} \): the first column contains the expansion coefficients of \( |\omega_1\rangle \) in that basis, the second column contains those of \( |\omega_2\rangle \) and so on.
Let’s check that it satisfies the column-wise and row-wise orthonormality conditions required of a unitary operator’s matrix representation. First, the column-wise proof:

$$\langle \text{column } j | \text{column } k \rangle = \sum_{m=1}^{n} \langle m | \omega_j \rangle^* \langle m | \omega_k \rangle = \sum_{m=1}^{n} \langle \omega_j | m \rangle \langle m | \omega_k \rangle = \langle \omega_j | \omega_k \rangle = \delta_{jk}$$

where we used the fact that the \{\ket{j}\} are an orthonormal basis that span the space and that the \{\ket{\omega_j}\} are an orthonormal set. Similarly, for the row-wise condition:

$$\langle \text{row } j | \text{row } k \rangle = \sum_{m=1}^{n} \langle j | \omega_m \rangle^* \langle k | \omega_m \rangle = \sum_{m=1}^{n} \langle k | \omega_m \rangle \langle \omega_m | j \rangle = \langle k | j \rangle = \delta_{kj}$$

where now we use the fact that the \{\ket{\omega_j}\} are an orthonormal basis that span the space and that the \{\ket{j}\} are an orthonormal set.
What does this unitary operator do? If we act on the column matrix representation of one of the basis elements that defines the matrix representation, we find it gets transformed into one of the eigenvectors; for example, acting on $|n\rangle$:

$$U_\Omega |n\rangle \leftrightarrow \begin{bmatrix} \langle 1 | \omega_1 \rangle & \cdots & \langle 1 | \omega_n \rangle \\ \vdots & \ddots & \vdots \\ \langle n | \omega_1 \rangle & \cdots & \langle n | \omega_n \rangle \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \langle 1 | \omega_n \rangle \\ \vdots \\ \langle n | \omega_n \rangle \end{bmatrix} = (3.76)$$

$$\leftrightarrow \sum_{j=1}^{n} \langle j | \omega_n \rangle |j\rangle = \sum_{j=1}^{n} |j\rangle \langle j | \omega_n \rangle = |\omega_n\rangle = (3.77)$$

where we again used the fact that the $\{|j\rangle\}$ are an orthonormal basis for the space to collapse the sum over $j$. Similarly, the reverse transformation from the eigenvectors to the original basis is performed by $U_\Omega^\dagger$; we summarize these two statements as

$$|\omega_j\rangle = U_\Omega |j\rangle \quad \quad |j\rangle = U_\Omega^\dagger |\omega_j\rangle = (3.78)$$
If $U_\Omega$ rotates the original orthonormal basis $\{|j\rangle\}$ to become the eigenvectors $\{|\omega_j\rangle\}$, and $U_\Omega^\dagger$ rotates the eigenvectors $\{|\omega_j\rangle\}$ to become the original orthonormal basis $\{|j\rangle\}$, we are led to the question: how does $U_\Omega$ act on the operator $\Omega$ that gave the eigenvectors? Consider the following:

$$\langle \omega_j | \Omega | \omega_k \rangle = \langle \omega_j | U_\Omega U_\Omega^\dagger \Omega \Omega | U_\Omega U_\Omega^\dagger | \omega_k \rangle = \langle j | U_\Omega^\dagger \Omega \Omega | k \rangle$$  \hspace{1cm} (3.79)

Since we know $\langle \omega_j | \Omega | \omega_k \rangle = \omega_j \delta_{jk}$, it must therefore hold that the unitary transformation

$$\Omega' = U_\Omega^\dagger \Omega U_\Omega$$  \hspace{1cm} (3.80)

gives a new operator $\Omega'$ that is diagonal in the original basis $\{|j\rangle\}$ and has the same eigenvalues, in the same order, as $\Omega$:

$$\langle \omega_j | \Omega | \omega_k \rangle = \langle j | \Omega' | k \rangle$$  \hspace{1cm} (3.81)
More generally, given an operator \( \Lambda \) and the unitary operator \( U_\Omega \), we define the transformed version \( \Lambda' \) in the same manner

\[
\Lambda' = U_\Omega^\dagger \Lambda U_\Omega \tag{3.82}
\]

Whether or not this transformation gives an operator \( \Lambda' \) that is diagonal in the original basis \( \{ |j\rangle \} \) like \( \Omega' \) depends on \( \Lambda \); we will return to this question soon.
We must explain a subtle point about “transformations” versus “arithmetic operations.” The above discussion gives us a new operator \( \Omega' = U^\dagger_\Omega \Omega U_\Omega \) that is different from \( \Omega \)!

The original operator \( \Omega \) is diagonal if one’s matrix representation uses the \( \{ |\omega_j\rangle \} \) as the basis; the new operator is diagonal if one’s matrix representation uses the original \( \{ |j\rangle \} \) basis. They are different operators and they have different eigenvectors.

One thing that is confusing about this is that the operators \( \Omega \) and \( \Omega' \) have the same eigenvalues and thus their diagonal forms are the same. One is tempted to think that they are the same operator. But, because they are diagonal in different matrix representations, they are most definitely not the same operator. An explicit way to see this is to write them out in the form given in Equation 3.72

\[
\Omega = \sum_{j=1}^{n} \omega_j |\omega_j\rangle \langle \omega_j | \quad \Omega' = \sum_{j=1}^{n} \omega_j |j\rangle \langle j | .
\]

The two forms involve outer products of entirely different sets of vectors, so they are different operators; it is only the coefficients that are the same.
We can see that each operator is diagonal in its own eigenbasis and not diagonal in the other’s eigenbasis by writing out the relevant matrix elements:

\[
\langle j | \Omega | k \rangle = \langle j | \left[ \sum_{m=1}^{n} \omega_m | \omega_m \rangle \langle \omega_m | \right] | k \rangle = \left[ \sum_{m=1}^{n} \omega_m \langle j | \omega_m \rangle \langle \omega_m | k \rangle \right]
\]

\[
\langle \omega_j | \Omega | \omega_k \rangle = \langle \omega_j | \left[ \sum_{m=1}^{n} \omega_m | \omega_m \rangle \langle \omega_m | \right] | \omega_k \rangle = \left[ \sum_{m=1}^{n} \omega_m \langle \omega_j | \omega_m \rangle \langle \omega_m | \omega_k \rangle \right]
\]

\[
\langle j | \Omega' | k \rangle = \langle j | \left[ \sum_{m=1}^{n} \omega_m | m \rangle \langle m | \right] | k \rangle = \omega_j \delta_{jk}
\]

\[
\langle \omega_j | \Omega' | \omega_k \rangle = \omega_j \delta_{jk} = \langle j | \Omega' | k \rangle
\]

The matrix elements are diagonal for each operator in its own eigenbasis, but are not necessarily diagonal for each operator in the other operator’s eigenbasis. We also see we recover Equation 3.83,
There is something that muddles all of this. To obtain the representation of an arbitrary operator $\Lambda$ in the matrix representation corresponding to the eigenbasis of $\Omega$, $\{|\omega_j\rangle\}$, we find we must apply the unitary transformation operation $U^\dagger \Lambda U$ to the matrix representation of $\Lambda$ in the original basis $\{|j\rangle\}$ as a purely arithmetic procedure. At the cost of some notational complexity, we can clarify the similarity and difference between the unitary transformation as an operator transformation, yielding a new operator $\Lambda'$, and its use as an arithmetic procedure to obtain a new matrix representation of the same operator $\Lambda$. Let us use the following notation:

- $\Lambda$ = an operator on a vector space, representation-free
- $[\Lambda]_j$ = the matrix representation of the operator $\Lambda$ in the $\{|j\rangle\}$ matrix representation
- $[\Lambda]_{\omega_j}$ = the matrix representation of the operator $\Lambda$ in the $\{|\omega_j\rangle\}$ matrix representation
Next, note the following relationship between matrix elements in the two different bases:

\[
\langle \omega_j | \Lambda | \omega_k \rangle = \sum_{p,q=1}^{n} \langle \omega_j | p \rangle \langle p | \Lambda | q \rangle \langle q | \omega_k \rangle = \sum_{p,q=1}^{n} \left[ U_\Omega^\dagger \right]_{jp} \langle p | \Lambda | q \rangle [U_\Omega]_{qk}
\]

So, the matrix elements are related by an arithmetic operation that is the same as the unitary transformation. Using our notation,

\[
\left[ \begin{array}{c} \Lambda \\ \end{array} \right] |_{\omega_j} = \left[ \begin{array}{c} \left[ U_\Omega^\dagger \right]_{|j} \\ \end{array} \right] \left[ \begin{array}{c} \Lambda \\ \end{array} \right] |_{j} = \left[ U_\Omega \right] |_{j} \tag{3.83}
\]

But we also have, based on \( \Lambda' = U_\Omega^\dagger \Lambda U_\Omega \):

\[
\left[ \begin{array}{c} \Lambda' \\ \end{array} \right] |_{j} = \left[ \begin{array}{c} \left[ U_\Omega^\dagger \right]_{|j} \\ \end{array} \right] \left[ \begin{array}{c} \Lambda \\ \end{array} \right] |_{j} = \left[ U_\Omega \right] |_{j} \tag{3.84}
\]
We may therefore state

\[ [ \Lambda']_j = [ \Lambda]_{\omega j} \]  \hspace{1cm} (3.85)

which is no doubt incredibly confusing: the matrix representation of the unitary-transformed operator \( \Lambda' \) in the original \( \{|j\rangle\} \) basis is the same as the matrix representation of the untransformed operator \( \Lambda \) in the eigenbasis \( \{|\omega_j\rangle\} \). Thus, one has to be very careful to understand from context whether one is staying in the same basis and transforming the operators or whether one is going to the matrix representation of the eigenbasis. The matrix representations will look the same! We usually want to do the latter, but in practice do the former.
Going back to Equation 3.83, let’s consider another possible source of confusion. Is it completely clear what we mean by $U_\Omega$ and $[\ U_\Omega\ ]_j$? We defined $U_\Omega$ by its matrix representation in the $\{|j\rangle\}$ basis, and that is what we mean above by $[\ U_\Omega\ ]_j$ and its adjoint. That’s clear and unambiguous.

But confusion may arise when one asks the obvious follow-on question to: what should the matrix representation of $U_\Omega$ in $\Omega$’s eigenbasis, $[\ U_\Omega\ ]_\omega$, be? Should it be the identity matrix because no transformation is needed if one’s matrix representation is already in the eigenbasis of $\Omega$? Applying Equation 3.83, we obtain

$$[\ U_\Omega\ ]_\omega = [\ U_\Omega^\dagger\ ]_j [\ U_\Omega\ ]_j [\ U_\Omega\ ]_j = [\ U_\Omega\ ]_j$$

which indicates that $U_\Omega$ has the same matrix representation in the two bases. Which is correct: is $U_\Omega$’s matrix representation independent of basis, or should $U_\Omega$ become the identity matrix in the eigenbasis of $\Omega$?

The confusion arises because we have been a bit ambiguous about what is meant by $U_\Omega$. $U_\Omega$ is the operator that transforms $\{|j\rangle\}$ into $\{|\omega_j\rangle\}$. This depends on both $\{|j\rangle\}$ and $\{|\omega_j\rangle\}$, not just on $\{|\omega_j\rangle\}$ (and thus not just on $\Omega$). Really, we ought to label $U_\Omega$ as $U_{|j\rangle\rightarrow|\omega_j\rangle}$ because $U$ is defined in terms of $\{|j\rangle\}$ and $\{|\omega_j\rangle\}$; it depends only indirectly on $\Omega$ through the fact that $\Omega$ determines what the $\{|\omega_j\rangle\}$ are. If one’s basis is already the eigenbasis of $\Omega$, then the unitary operator one wants is $U_{|\omega_j\rangle\rightarrow|\omega_j\rangle} = I$. That is a different operator from $U_{|j\rangle\rightarrow|\omega_j\rangle}$.
Thus, strictly speaking, the preceding equation that relates the matrix elements of $U_\Omega$ in the two bases should be written as

$$
\begin{bmatrix}
U|j\rangle\rightarrow|\omega_j\rangle
\end{bmatrix}_{|\omega_j\rangle} =
\begin{bmatrix}
U^\dagger|j\rangle\rightarrow|\omega_j\rangle
\end{bmatrix}_{|j\rangle} \begin{bmatrix}
U|j\rangle\rightarrow|\omega_j\rangle
\end{bmatrix}_{|j\rangle} =
\begin{bmatrix}
U|j\rangle\rightarrow|\omega_j\rangle
\end{bmatrix}_{|j\rangle}
$$

The matrix representation of $U|j\rangle\rightarrow|\omega_j\rangle$ is indeed unchanged by the unitary transformation. The resolution of the misconception is that this is no longer the operator one wants: if working in the $|\omega_j\rangle$ basis, one wants $U|\omega_j\rangle\rightarrow|\omega_j\rangle = I$. The above form for $\begin{bmatrix}
U|j\rangle\rightarrow|\omega_j\rangle
\end{bmatrix}_{|\omega_j\rangle}$ is therefore not wrong, it is simply not useful. It now would rotate the eigenbasis of $\Omega$ to some new set of vectors in the space that are neither the original basis nor the eigenbasis of $\Omega$.

Clearly, there is much opportunity for confusion. We cannot use the above notation in general because it is too complicated to carry around. We will have to rely on context to understand which $U_\Omega$ we are interested in. One saving grace, though, is that, once we have decided which bases $U_\Omega$ will transform between, then its matrix representation is independent of basis choice. Therefore, we will not need to write $\begin{bmatrix}
U_\Omega
\end{bmatrix}_{|j\rangle}$, we can simply write $U_\Omega$. 