Lecture 29:
Multiparticle Systems
Direct Product Spaces: States
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Direct Product Spaces

Suppose we have two Hilbert spaces, \( V_1 \) and \( V_2 \), each containing the states corresponding to a particular degree of freedom (dof); a typical example is that \( V_1 \) contains the states for particle 1 and \( V_2 \) for particle 2, where both particles live in a single spatial dimension. Then we can build a new Hilbert space, \( V = V_1 \otimes V_2 \), that contains the state of the two particles considered together. This space is called a direct product space. Formally, we construct the elements of the space in three steps:

- First, we define elements that are combinations of single states from the two factor spaces:

\[
|v, w\rangle^{(1) \otimes (2)} \equiv |v\rangle^{(1)} \otimes |w\rangle^{(2)}
\]

where the superscripts on each ket outside the ket bracket indicates which particle’s Hilbert space it belongs to: the \( ^{(1)} \) kets belong to \( V_1 \), the \( ^{(2)} \) kets to \( V_2 \) and the \( ^{(1) \otimes (2)} \) kets to \( V = V_1 \otimes V_2 \). We emphasize that this definition cannot be algebraically reduced to something simpler. An example of the above would be for \( |v\rangle^{(1)} \) to be a basis element of the position-basis representation for particle 1, \( |x\rangle \), and for \( |w\rangle^{(2)} \) to be a basis element of the position-basis element for particle 2, \( |\tilde{x}\rangle \). The direct product vector

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|\tilde{x}, \tilde{x}\rangle^{(1)\otimes(2)} = |\tilde{x}\rangle^{(1)} \otimes |\tilde{x}\rangle^{(2)} is simply the state in which particle 1 is at \(\tilde{x}\) and particle 2 is at \(\tilde{x}\); it can be written in no simpler fashion.

Note that many authors would have written \(|x_1\rangle, |x_2\rangle and |x_1, x_2\rangle, dropping the superscripts altogether and relying on context to make it clear which state refers to particle 1, particle 2, and the combination of the two. There is nothing wrong with this, but the use of the numbered subscripts makes it seem that the position value \(x_1\) is only available to the first particle and the position value \(x_2\) only to the second particle. However, the particles live in the same physical space, both \(x_1\) and \(x_2\) are accessible to both, and it would be perfectly reasonable to have \(|x_2\rangle^{(1)}, |x_1\rangle^{(2)}, and |x_2, x_1\rangle^{(1)\otimes(2)} = |x_2\rangle^{(1)} \otimes |x_1\rangle^{(2)}.

Second, all possible linear combinations operate in the expected manner, assuming the two factor spaces \(V_1\) and \(V_2\) have the same field (complex numbers for what we usually consider):

\[
\left(\alpha |v\rangle^{(1)}\right) \otimes \left(\beta |w\rangle^{(2)}\right) = \alpha \beta |v, w\rangle^{(1)\otimes(2)}
\]
\[
\left(\alpha_1 |u_1\rangle^{(1)} + \alpha_2 |v_1\rangle^{(1)}\right) \otimes |w\rangle^{(2)} = \alpha_1 |u_1, w\rangle^{(1)\otimes(2)} + \alpha_2 |v_1, w\rangle^{(1)\otimes(2)}
\]
\[
|u\rangle^{(1)} \otimes \left(\beta_1 |v_2\rangle^{(2)} + \beta_2 |w_2\rangle^{(2)}\right) = \beta_1 |u, v_2\rangle^{(1)\otimes(2)} + \beta_2 |u, w_2\rangle^{(1)\otimes(2)}
\]
Third, the inner product is defined as the obvious extension of the inner product in each space:

\[
(1) \otimes (2) \langle v_1, w_1 | v_2, w_2 \rangle^{(1) \otimes (2)} \equiv \left( (1) \langle v_1 | v_2 \rangle^{(1)} \right) \left( (2) \langle w_1 | w_2 \rangle^{(2)} \right)
\]

By including linear combinations and requiring the scalar fields of the two products spaces be the same, we ensure that the direct product space is a vector space. A reasonable basis for the product space is simply the set of direct products of the bases of the individual spaces; that is, suppose \( \{ |n\rangle^{(1)} \} \) are a basis for the first space and \( \{ |m\rangle^{(2)} \} \) are a basis for the second space. Then a basis for the direct product space consists of all products of the form

\[
|n, m\rangle^{(1) \otimes (2)} = |n\rangle^{(1)} \otimes |m\rangle^{(2)}
\]

where both \( n \) and \( m \) run over their full range of values. If the two factor spaces have dimension \( N \) and \( M \) (with these possibly being infinite), then the direct product space has dimension \( N \times M \).
By defining the inner product as above, the direct product space inherits all the necessary inner product properties from the factor spaces, rendering the direct product space an inner product space.

Finally, the restriction to normalizable states that occurs to render the inner product space into the physical Hilbert space is inherited via the inherited definition of inner product. So we are automatically assured that all elements of the direct product space are normalizable if the factor-spaces are physical Hilbert spaces.

Note that the fact that the direct product space is a physical Hilbert space ensures that Postulate 1 continues to be satisfied.
The Null Vector, Inverse Vectors, Invertibility, and Entanglement

The null vector and inverse vectors are a bit tricky in direct product spaces because there are multiple ways to construct them. First, any direct product in which one factor is a null vector from either space gives the null vector of the direct product space:

\[ |0\rangle^{(1)} \otimes |w\rangle^{(2)} = |0\rangle^{(1) \otimes (2)} = |v\rangle^{(1)} \otimes |0\rangle^{(2)} \]

We can see that the two factor forms are equivalent by calculating their norms: in each case, the norm vanishes because the norm of the direct product is the product of the norms, and one factor has vanishing norm in either case. The definition of inner product requires that the null vector be the only vector that has vanishing norm, so we must take as a definition that all these ways of obtaining \( |0\rangle^{(1) \otimes (2)} \) are equivalent in order for the direct product space to be an inner product (and hence physical Hilbert) space.

An implication of this is that the mapping from the two factor spaces to the direct product space is not *one-to-one* and hence is *noninvertible*.
The same issue arises for inverses. There are multiple pairs in the factor spaces that map to the inverse of a given member of the direct product space:

\[-|v, w\rangle^{(1)\otimes(2)} = - \left(|v\rangle^{(1)} \otimes |w\rangle^{(2)}\right)\]

\[= \left(-|v\rangle^{(1)}\right) \otimes |w\rangle^{(2)} = |v\rangle^{(1)} \otimes \left(-|w\rangle^{(2)}\right)\]

We can see another way in which the mapping from factor space pairs to the direct product space is noninvertible: in addition to not being one-to-one, it is also not onto. That is, not every element of the direct product space can be written purely as a product of elements of the factor spaces. A simple example can be construction from any two basis elements of the direct product space:

\[|n_1\rangle^{(1)} \otimes |m_1\rangle^{(2)} + |n_2\rangle^{(1)} \otimes |m_2\rangle^{(2)} \neq |v\rangle^{(1)} \otimes |w\rangle^{(2)}\]

for all $|v\rangle^{(1)}$ in $\mathbb{V}_1$ and all $|w\rangle^{(2)}$ in $\mathbb{V}_2$

It is easy to prove this by assuming that the above is true, expanding the $|v\rangle^{(1)}$ in terms of the $\{|n\rangle^{(1)}\}$ and $|w\rangle^{(2)}$ in terms of the $\{|m\rangle^{(2)}\}$, and obtaining a contradiction. We will do this later.
States of the above type are called entangled – neither degree of freedom (think “particle”) is in a particular state in its own space! This is a fundamentally quantum mechanical phenomenon that arises from the fact that the state of a particle is represented by vectors in a Hilbert space and that these Hilbert spaces can be direct producted together in a noninvertible manner. Entanglement, which arises from the noninvertibility, makes the physics of systems with multiple degrees of freedom more than just the some of the parts.
Direct Products vs. Direct Sums

Recall that we defined direct sums early on in the course in connection with the idea of vector subspaces: a direct sum space $V_1 \oplus V_2$ consists of all linear combinations of elements of the two spaces $V_1$ and $V_2$. Since it is required that one be able to add elements of $V_1$ and $V_2$, they must already be subspaces of a larger vector space $V$: that is, elements of $V_1$ and of $V_2$ already belong to $V$ and there is already a rule for how to add them.

A direct product space is quite different. Perhaps the most interesting difference is the fact that that the direct product space $V_1 \otimes V_2$ requires no prior existence of a vector space $V$ in which $V_1 \otimes V_2$ is contained. That is, $V_1 \otimes V_2$ is a new vector space that is in no way reducible in terms of $V_1$ and $V_2$ separately. Of course, $V_1 \otimes V_2$ may look like a space we have already seen in some cases, but that is not a generic statement. In general, $V_1 \otimes V_2$ is just a new object.

One specific technical difference between direct sum and direct product spaces is that the construction of the former is invertible while that of the latter is not. Any element of $V_1 \oplus V_2$ can be written uniquely as a sum of elements of $V_1$ and $V_2$: just decompose it in terms of the basis of $V_1 \oplus V_2$ and split up the terms into basis elements belonging to $V_1$ and to $V_2$. As we explained above, such a decomposition is not in general possible for elements of a direct product space.
Pitfalls in Understanding Direct Product Spaces

In the case of taking the direct product of the Hilbert spaces of two particles that live in the same spatial space, a typical pitfall is the desire to put the particles in the same Hilbert space. Be wary of this! Except through explicit interactions (which we will encounter later) of the two particles, *the Hilbert space states of the two particles are totally independent* even though, when one projects onto, say, the position basis, both wavefunctions give the probability of detecting the two particles in the same physical space and may overlap.

Another possible stumbling block: don’t confuse the Hilbert space dimension with the dimension of the physical space that the particles live in; we will give examples below.
Examples of Direct Product Spaces

As we have indicated above, one can construct a direct product Hilbert space from multiple single-dof Hilbert spaces. For the free particle, each factor space has dimension equal to the size of the real numbers, so the product space has that size squared (which is the same). For the particle in a box or SHO, the number of single-dof states is countably infinite; the square of that number is also countably infinite. In both cases, the two particles move about in a single, shared spatial dimension, and the number of degrees of spatial freedom is two: the spatial coordinates of the two particles.

Different spatial degrees of freedom of a single particle can be put together via a direct product to give the full two- or three-spatial-dimensional state of that particle.

One can combine spatial and other degrees of freedom. For a hydrogen atom, one factor space would consist of the three-spatial-dimensional center-of-mass position, while the other would consist of the three-spatial-dimensional relative electron-proton position (described by the radial quantum number $n$ and the angular momentum quantum numbers $l^2$ and $l_z$, which we will cover in detail later). Another example would be a rigid rotator, where again one factor space is the center-of-mass position and the other is the same $l^2$ and $l_z$ quantum numbers (there is no radial quantum number because the body is rigid).

One can of course combine multiple degrees of freedom for multiple particles.