Physics 125a – Problem Set 1 – Due Oct 8, 2007
Solutions

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Problem 1

We will first provide the solutions for vector spaces that are "well-behaved". A few students pointed out that, if one assumes that vector space addition and scalar-vector multiplication are not well-behaved – *e.g.*, $\alpha|v\rangle = |0\rangle$ for $\alpha \neq 0$ – then the proofs fall apart. We'll address this point below.

(a) Consider an arbitrary vector $|v\rangle$ and arbitary scalar coefficients α and β. Then

$$(\alpha+\beta)\,|v\rangle = \alpha\,|v\rangle + \beta\,|v\rangle = \beta\,|v\rangle + \alpha\,|v\rangle = (\beta+\alpha)\,|v\rangle$$

Since this relation holds for arbitary $|v\rangle$, it must be that $\alpha+\beta = \beta+\alpha$. Similarly,

$$\begin{aligned}(\alpha+(\beta+\gamma))\,|v\rangle &= \alpha\,|v\rangle + (\beta+\gamma)\,|v\rangle = \alpha\,|v\rangle + (\beta\,|v\rangle + \gamma\,|v\rangle) = (\alpha\,|v\rangle + \beta\,|v\rangle) + \gamma\,|v\rangle \\ &= (\alpha+\beta)\,|v\rangle + \gamma\,|v\rangle = ((\alpha+\beta)+\gamma)\,|v\rangle\end{aligned}$$

Again, since the relation holds for arbitrary $|v\rangle$, it must hold that $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$

(b) Consider an arbitrary vector $|v\rangle$ and arbitrary scalar coefficients α, β, and γ. Then

$$(\alpha\,(\beta\gamma))\,|v\rangle = \alpha\,((\beta\gamma)\,|v\rangle) = \alpha\,(\beta\gamma\,|v\rangle) = \alpha\beta\gamma\,|v\rangle = (\alpha\beta)\,(\gamma\,|v\rangle) = ((\alpha\beta)\,\gamma)\,|v\rangle$$

By the same argument as before, it must be true that $\alpha\,(\beta\gamma) = (\alpha\beta)\,\gamma$.

(c) Consider an arbitrary vector $|v\rangle$. It holds that

$$(-1)\,|v\rangle + |v\rangle = (-1+1)\,|v\rangle = 0|v\rangle$$

In order to complete the proof, we need to show that $0|v\rangle = |0\rangle$, because then it will hold that $(-1)\,|v\rangle + |v\rangle = |0\rangle$; since additive inverses in the vector space are unique, it thus holds that $(-1)\,|v\rangle = -|v\rangle$

To prove $0|v\rangle = |0\rangle$, consider an arbitrary vector $|v\rangle$ and an arbitrary scalar coefficient α. Then

$$0|v\rangle + \alpha|v\rangle = (0+\alpha)\,|v\rangle = \alpha|v\rangle$$

However, $|0\rangle + \alpha|v\rangle = \alpha|v\rangle$ because $|0\rangle$ is the additive identity in the vector space. We stipulated in the definition a vector space that the vector space addition identity element is unique, so it must hold that $0|v\rangle = |0\rangle$.

Hence the proof for the action of -1 goes through.

We made implicit assumptions in the above proofs. In all three, we assumed that, if $\alpha|v\,\rangle = \beta|v\,\rangle$ for any $|v\,\rangle$, then $\alpha = \beta$ must hold. In (c), we assumed that $\alpha|v\,\rangle$ was not already $|0\,\rangle$.

A few students pointed out that these assumptions may break down. For example, suppose that we define that all scalar multiplication yields $|0\,\rangle$: $\alpha|v\,\rangle = |0\,\rangle$ for all α and all $|v\,\rangle$. Such a rule would void the proofs for (a) and (b) because the arbitrariness of $|v\,\rangle$ in the proof does not then imply that $\alpha = \beta$ is necessary: α and β could be anything, and $|v\,\rangle$ could be anything, yet the equality $\alpha|v\,\rangle = \beta|v\,\rangle$ would hold because $\alpha|v\,\rangle = |0\,\rangle = \beta|v\,\rangle$.

It was claimed that another counterexample is a case in which $|v\,\rangle + |w\,\rangle = |0\,\rangle$ for $|w\,\rangle \neq -|v\,\rangle$ for some pairs $|v\,\rangle$, $|w\,\rangle$. This is not a valid counterexample because it gives $|v\,\rangle$ a nonunique inverse, which violates our vector space definitions.

The only defense against such counterexamples is that vector spaces for which the proofs do not go through in the given manner would be of no physical interest. To be safe, the vector space definition in the notes should be augmented to require that the scalar-vector multiplication rule be nontrivial. But this isn't a math class, so we'll just drop the issue at this point.

We'll give full credit to anyone who either provided the above standard proofs under the "well-behaved vector space" assumption or provided a counterexample to demonstrate that these theorems were invalid without such an assumption.

Problem 2

For an antisymmetric matrix A, $A_{ij} = -A_{ji}$. Therefore each diagonal elememt is zero ($A_{ii} = -A_{ii} = 0$). For 3×3 matrix $|C\rangle$, it has the following form.

$$|C\rangle = \begin{bmatrix} 0 & C_1 & C_2 \\ -C_1 & 0 & C_3 \\ -C_2 & -C_3 & 0 \end{bmatrix}$$

Now for $|C\rangle$ to be linearly independent from $|1\rangle$ and $|2\rangle$, if $\alpha\,|1\rangle + \beta\,|2\rangle\; + \gamma\,|C\rangle = 0$, our definition of linear independence states that $\alpha = \beta = \gamma = 0$ should be the only possibility:

$$\alpha\,|1\rangle + \beta\,|2\rangle\; + \gamma\,|C\rangle = \begin{bmatrix} 0 & \beta + \gamma C_1 & \alpha + \gamma C_2 \\ -\left(\beta + \gamma C_1\right) & 0 & \gamma C_3 \\ -\left(\alpha + \gamma C_2\right) & -\gamma C_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In other words, the following equations

$$\beta + \gamma C_1 \;= 0 \tag{1}$$
$$\alpha + \gamma C_2 \;= 0 \tag{2}$$
$$\gamma C_3 \;= 0 \tag{3}$$

should be true only when $\alpha = \beta = \gamma = 0$. If $C_3 = 0$, then γ does not have to be zero for Equation 3 to be true. Therefore C_3 should not be zero. Then γ must be zero and α and β are zero from Equation 1 and Equation 2. Thus every $|C\rangle$ whose C_3 is not 0 is linearly independent from $|1\rangle$ and $|2\rangle$. But $|C\rangle$ may not be orthogonal to $|1\rangle$ and $|2\rangle$. For example, if

$$|C\rangle = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

then $|C\rangle$ is linearly independent but not orthogonal to $|1\rangle$ and $|2\rangle$, as you can verify by a direct calculation.

Now let's find a vector $|C'\rangle$ from $|C\rangle$ that is orthogonal to $|1\rangle$ and $|2\rangle$ by Gram-Schmidt orthogonalization:

$$\left|C'\right\rangle = |C\rangle - \frac{\langle 1|C\rangle}{\langle 1|1\rangle}|1\rangle - \frac{\langle 2|C\rangle}{\langle 2|2\rangle}|2\rangle = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & C_3 \\ 0 & -C_3 & 0 \end{bmatrix}$$

$|C'\rangle$ is a scalar multiple of $|3\rangle$, or, is parallel to $|3\rangle$. We see that, once orthonormality is specified, the freedom in the third vector vanishes.

Problem 3

Let's find out whether the given operators are hermitian, antihermitian or neither.

$$\begin{aligned} (\Omega\Lambda)^\dagger &= \Lambda^\dagger\Omega^\dagger = \Lambda\Omega && \therefore \text{ neither.} \\ (\Omega\Lambda + \Lambda\Omega)^\dagger &= \Lambda^\dagger\Omega^\dagger + \Omega^\dagger\Lambda^\dagger = \Lambda\Omega + \Omega\Lambda = \Omega\Lambda + \Lambda\Omega && \therefore \text{ hermitian.} \\ [\Omega, \Lambda]^\dagger &= (\Omega\Lambda - \Lambda\Omega)^\dagger = \Lambda^\dagger\Omega^\dagger - \Omega^\dagger\Lambda^\dagger = \Lambda\Omega - \Omega\Lambda = -(\Omega\Lambda - \Lambda\Omega) \\ &= -[\Omega, \Lambda] && \therefore \text{ antihermitian.} \\ (i[\Omega, \Lambda])^\dagger &= -i[\Omega, \Lambda]^\dagger = i[\Omega, \Lambda] && \therefore \text{ hermitian.} \end{aligned}$$

The above combinations will be useful in constructing Hermitian operators for physical quantities that are products of other physical quantities.

Problem 4

(a) If U is a unitary matrix, $U^\dagger U = I$. Then

$$\begin{aligned} \det\left(U^\dagger U\right) &= \det\left(U^\dagger\right)\det(U) = \det\left(\left(U^T\right)^*\right)\det(U) = \left(\det\left(U^T\right)\right)^*\det(U) \\ &= (\det(U))^*\det(U) = \|\det(U)\|^2 = \det(I) = 1 \end{aligned}$$

Because the modulus of a complex number is real, $\|\det(U)\| = 1$.

If we make a unitary transformation of matrix A using U, the result is $U^\dagger AU$. Then

$$\begin{aligned} \det\left(U^\dagger AU\right) &= \det\left(U^\dagger\right)\det(A)\det(U) = \det(A)\det(U)\det\left(U^\dagger\right) \\ &= \det\left(AUU^\dagger\right) = \det(A) \end{aligned}$$

(b)

$$\mathrm{Tr}(\Omega\Lambda) = \sum_{i,j}\Omega_{ij}\Lambda_{ji} = \sum_{i,j}\Lambda_{ji}\Omega_{ij} = \mathrm{Tr}(\Lambda\Omega)$$

$$\mathrm{Tr}(\Omega\Lambda\Theta) = \sum_{i,j,k}\Omega_{ij}\Lambda_{jk}\Theta_{ki} = \sum_{i,j,k}\Lambda_{jk}\Theta_{ki}\Omega_{ij} = \mathrm{Tr}(\Lambda\Theta\Omega) = \sum_{i,j,k}\Theta_{ki}\Omega_{ij}\Lambda_{jk} = \mathrm{Tr}(\Theta\Omega\Lambda)$$

$$\mathrm{Tr}\left(U^\dagger\Omega U\right) = \mathrm{Tr}\left(\Omega UU^\dagger\right) = \mathrm{Tr}(\Omega I) = \mathrm{Tr}(\Omega)$$