Physics 125a – Problem Set 2 – Due Oct 21, 2008
Solutions
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Problem 1

According to example 3.23, we know the operator operating on the vector space can be defined as

\[ \hat{S}_z \ket{\uparrow} = \frac{\hbar}{2} \ket{\uparrow} \]

\[ \hat{S}_z \ket{\downarrow} = -\frac{\hbar}{2} \ket{\downarrow} \]

Now let’s assume the operator operating on the dual vector spaces can be written in terms of linear combination of dual vectors with some undetermined factors.

\[ \bra{\uparrow} \hat{S}_z = \bra{\uparrow} \alpha_{11} + \bra{\downarrow} \alpha_{12} \]

\[ \bra{\downarrow} \hat{S}_z = \bra{\uparrow} \alpha_{21} + \bra{\downarrow} \alpha_{22} \]

Then we can use the inner products to determine the values of \( \alpha \)’s. The four inner products are

\[ \bra{\uparrow} \ket{\hat{S}_z} = \bra{\uparrow} \frac{\hbar}{2} \]

\[ \bra{\downarrow} \ket{\hat{S}_z} = \bra{\downarrow} (\frac{\hbar}{2}) \]

Thus we can see that

\[ \bra{\uparrow} \ket{\hat{S}_z} = \bra{\uparrow} \frac{\hbar}{2} \]

\[ \bra{\downarrow} \ket{\hat{S}_z} = \bra{\downarrow} (-\frac{\hbar}{2}) \]

Above I used the orthogonality condition

\[ \bra{\uparrow} \ket{\downarrow} = 0 \]

Thus we can see that action of the operator \( \hat{S}_z \) on a vector space along with the definition of inner product fully specifies the action of the operator \( \hat{S}_z \) on the corresponding dual vector space. In fact, if you know that the operator \( \hat{S}_z \) is hermitian, then the relation between the operator \( \hat{S}_z \) on vector space and on dual vector space is trivial as shown below

\[ (\hat{S}_z \ket{\uparrow})^\dagger = \bra{\uparrow} \hat{S}_z^\dagger = \bra{\uparrow} \hat{S}_z \]

\[ (\hat{S}_z \ket{\downarrow})^\dagger = \bra{\downarrow} \hat{S}_z^\dagger = \bra{\downarrow} \hat{S}_z \]
Problem 2

In the beginning, for simplicity, let’s define

\[
|1\rangle \equiv \frac{|\hat{x}\rangle + |\hat{y}\rangle}{\sqrt{2}} \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad |2\rangle \equiv \frac{|\hat{x}\rangle - |\hat{y}\rangle}{\sqrt{2}} \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad |3\rangle \equiv |\hat{z}\rangle \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\] (14)

where the symbol \( \leftrightarrow \) means "represented by". In this way, then we can invert the equation to get

\[
|\hat{x}\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}} \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad |\hat{y}\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}} \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad |\hat{z}\rangle = |3\rangle \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\] (15)

Then, from the equations above, it’s easy to write down the matrix representations of the projection operators

\[
P_x \leftrightarrow |\hat{x}\rangle\langle \hat{x}| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\] (16)

\[
P_y = |\hat{y}\rangle\langle \hat{y}| \leftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\] (17)

\[
P_z = |\hat{z}\rangle\langle \hat{z}| \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\] (18)

\[
P_{x\hat{y}} = |\hat{x}\rangle\langle \hat{x}| + |\hat{y}\rangle\langle \hat{y}| = P_x + P_y = 1 - P_z \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\] (19)

\[
P_{y\hat{z}} = |\hat{y}\rangle\langle \hat{y}| + |\hat{z}\rangle\langle \hat{z}| = P_y + P_z = 1 - P_x \leftrightarrow \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\] (20)

\[
P_{x\hat{z}} = |\hat{x}\rangle\langle \hat{x}| + |\hat{z}\rangle\langle \hat{z}| = P_x + P_z = 1 - P_y \leftrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}
\] (21)

\[
P_{(\hat{x}+\hat{y})/\sqrt{2}} = \left( \frac{|\hat{x}\rangle + |\hat{y}\rangle}{\sqrt{2}} \right) \left( \frac{|\hat{x}\rangle + |\hat{y}\rangle}{\sqrt{2}} \right) \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\] (22)

\[
P_{(\hat{x}-\hat{y})/\sqrt{2}} = \left( \frac{|\hat{x}\rangle - |\hat{y}\rangle}{\sqrt{2}} \right) \left( \frac{|\hat{x}\rangle - |\hat{y}\rangle}{\sqrt{2}} \right) \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\] (23)

where above I used the identity that

\[
P_x + P_y + P_z = 1
\] (24)
Problem 3

(a) Let’s conjugate $\Omega \Lambda$

$$ (\Omega \Lambda)^\dagger = \Lambda^\dagger \Omega^\dagger = \Lambda \Omega $$

(25)

So we see that for general operators $\Omega$ and $\Lambda$ there is not much to say about the properties of their product under conjugation, but we can say something in special cases. For example, if the two operators are commuting, then their product is a Hermitian operator. If they are anticommuting, then their product is an anti-Hermitian operator.

(b) $$(\Omega \Lambda + \Lambda \Omega)^\dagger = \Lambda^\dagger \Omega^\dagger + \Omega^\dagger \Lambda^\dagger = \Lambda \Omega + \Omega \Lambda = \Omega \Lambda + \Lambda \Omega$$

(26)

This operator is Hermitian.

(c) $$[\Omega, \Lambda]^\dagger = (\Omega \Lambda - \Lambda \Omega)^\dagger = \Lambda^\dagger \Omega^\dagger - \Omega^\dagger \Lambda^\dagger = \Lambda \Omega - \Omega \Lambda = -[\Omega, \Lambda]$$

(27)

This operator is anti-Hermitian.

(d) $$(i[\Omega, \Lambda])^\dagger = -i(\Omega \Lambda - \Lambda \Omega)^\dagger = -i(\Lambda^\dagger \Omega^\dagger - \Omega^\dagger \Lambda^\dagger) = -i(\Lambda \Omega - \Omega \Lambda) = i[\Omega, \Lambda]$$

(28)

This operator is Hermitian.

Problem 4

(a) We can simply write down the matrices

$$ \Omega = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad \Omega^* = \begin{bmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{bmatrix} $$

(29)

Thus, it’s trivial to see that

$$ \text{Det}(\Omega^*) = \alpha^* \delta^* - \beta^* \gamma^* = [\text{Det}\Omega]^* $$

(30)

(b) Similarly, we can write down

$$ \Omega^T = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} $$

(31)

Thus,

$$ \text{Det}(\Omega^T) = \alpha \delta - \beta \gamma = \text{Det}(\Omega) $$

(32)

(c) Now $\Omega$ is Hermitian, which means

$$ \Omega = \Omega^\dagger = (\Omega^*)^T $$

(33)

Thus,

$$ \text{Det}(\Omega) = \text{Det}(\Omega^*)^T = \text{Det}(\Omega^*) = [\text{Det}(\Omega)]^* $$

(34)

Above I borrowed the result from (a) and (b). Therefore, $\text{Det}(\Omega)$ is real.
(d) Similarly, let’s write down the corresponding matrices
\[ \Lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Omega \Lambda = \begin{bmatrix} a \alpha + c \beta & b \alpha + d \beta \\ a \gamma + c \delta & b \gamma + d \delta \end{bmatrix} \] (35)

Thus, it’s straightforward to expand them to find that
\[ \text{Det}(\Omega \Lambda) = [\text{Det}(\Omega)][\text{Det}(\Lambda)] \] (36)

(e) Now \( \Omega \) is unitary, which means
\[ \Omega^\dagger \Omega = 1 \]
\[ \Rightarrow \text{Det}(\Omega^\dagger \Omega) = 1 \]
\[ \Rightarrow \text{Det}[(\Omega^*)^T]\text{Det}(\Omega) = 1 \]
\[ \Rightarrow \text{Det}(\Omega^*)\text{Det}(\Omega) = 1 \]
\[ \Rightarrow |\text{Det}(\Omega)|^2 = 1 \] (37)

Thus, \( \text{Det}\Omega \) is a complex number of unity modulus

(f) To solve the problem, let’s just keep using the results we got above
\[ \text{Det}(U^\dagger \Omega U) \]
\[ = \text{Det}(U^\dagger)\text{Det}(\Omega U) = \text{Det}(U^\dagger)\text{Det}(U)\text{Det}(\Omega) \]
\[ = \text{Det}(U^\dagger)\text{Det}(U)\text{Det}(\Omega) = \text{Det}(U^\dagger U)\text{Det}(\Omega) \]
\[ = \text{Det}(\Omega) \] (38)

Problem 5

\[ \text{Tr} (\Omega \Lambda) = \sum_{i,j} \Omega_{ij} \Lambda_{ji} = \sum_{i,j} \Lambda_{ji} \Omega_{ij} = \text{Tr} (\Lambda \Omega) \]
\[ \text{Tr} (\Omega \Lambda \Theta) = \sum_{i,j,k} \Omega_{ij} \Lambda_{jk} \Theta_{ki} = \sum_{i,j,k} \Lambda_{jk} \Theta_{ki} \Omega_{ij} = \text{Tr} (\Lambda \Theta \Omega) = \sum_{i,j,k} \Theta_{ki} \Omega_{ij} \Lambda_{jk} = \text{Tr} (\Theta \Omega \Lambda) \]
\[ \text{Tr} \left( U^\dagger \Omega U \right) = \text{Tr} \left( \Omega U U^\dagger \right) = \text{Tr} (\Omega I) = \text{Tr} (\Omega) \]

Problem 6

According to the notes, the action of the derivative operator on the space of functions on \( N \) discrete points is as follows:
\[ D_R |j \rangle = \left\{ \begin{array}{ll}
-|1\rangle - |N\rangle / \Delta & j = 1 \\
-|j\rangle - |j-1\rangle / \Delta & j \neq 1
\end{array} \right. \] (39)

We also have from the lecture notes the matrix elements:
\[ (D_R)_{kj} = \left\{ \begin{array}{ll}
(\delta_{k,N} - \delta_{k,1}) / \Delta & j = 1 \\
(\delta_{k,j-1} - \delta_{k,j}) / \Delta & j \neq 1
\end{array} \right. \] (40)
Thus the derivative operator $D_R$ has matrix representation for $N = 4$:

$$D_R = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$  \hspace{1cm} (41)$$

Thus, we can see that

$$(D_R)^\dagger = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \neq (D_R)$$  \hspace{1cm} (42)$$

Thus, $D_R$ is not Hermitian. And here we mention an important point. From class we know that the derivative operator $D_R$ gives us the “right-sided” discrete derivative of $f(x_j)$, and we should ask ourselves whether $D_R^\dagger$ also gives us some kind of discrete derivative of $f(x_j)$. The answer is “yes!” We can directly apply $D_R^\dagger$ to see this:

$$D_R^\dagger|j\rangle \leftrightarrow \frac{1}{\Delta} \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} f(x_4) - f(x_1) \\ f(x_1) - f(x_2) \\ f(x_2) - f(x_3) \\ f(x_3) - f(x_4) \end{bmatrix}$$  \hspace{1cm} (43)$$

In fact, the action of $D_R^\dagger$ the space of functions on $N$ discrete points is as follows

$$D_R^\dagger|j\rangle = \begin{cases} \frac{1}{\Delta}(|j+1\rangle - |j\rangle) & j \neq N \\ \frac{1}{\Delta}(|1\rangle - |N\rangle) & j = N \end{cases}$$  \hspace{1cm} (44)$$

Clearly, the action of the operator $D_R^\dagger$ on the space of functions on $N$ discrete points gives us the “left-sided” discrete derivative of $f(x_j)$, up to a sign. Therefore, the operator given in the problem $D_L$ is related to the operator $D_R^\dagger$ by

$$D_L = -D_R^\dagger$$  \hspace{1cm} (45)$$

Back to the question, how do we redefine $D_R$ and $D_L$ to obtain a Hermitian derivative operator? In quantum mechanics, the usual way to obtain a Hermitian operator from a non-Hermitian operator is to combine the original non-Hermitian operator with its adjoint. In this case, we can try to combine them to see the result:

$$\frac{D_R + D_R^\dagger}{2} = \frac{D_R - D_L}{2} = \frac{1}{2\Delta} \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}$$  \hspace{1cm} (46)$$

This operator is Hermitian, but it doesn’t actually give the discrete derivative anymore. Inspired by problem (3d), we instead combine them as follows:

$$D_H = i \frac{D_R - D_R^\dagger}{2} = i \frac{D_R + D_L}{2} = i \frac{1}{2\Delta} \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$  \hspace{1cm} (47)$$
So, we see that the new operator $D_H$ defined above is Hermitian. Its action on the vector $|f\rangle$ is as follows:

$$D_H|f\rangle \leftarrow \frac{i}{2\Delta} \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{bmatrix} = \frac{i}{2\Delta} \begin{bmatrix} f(x_2) - f(x_4) \\ f(x_3) - f(x_1) \\ f(x_4) - f(x_2) \\ f(x_1) - f(x_3) \end{bmatrix}$$

(48)

From this result, we can easily write down the general result for $N$ discrete points:

$$D_H|f\rangle = D_H \sum_{j=1}^{N} f(x_j)|j\rangle$$

$$= i \left[ \sum_{j=2}^{N-1} \frac{f(x_{j+1}) - f(x_j) + f(x_j) - f(x_{j-1})}{2\Delta} |j\rangle \right]$$

$$+ i \left[ \frac{f(x_2) - f(x_1) + f(x_1) - f(x_N)}{2\Delta} |1\rangle + \frac{f(x_1) - f(x_N) + f(x_N) - f(x_{N-1})}{2\Delta} |N\rangle \right]$$

(49)

Thus, we can see that the action of the Hermitian operator $D_H$ on the space of functions on $N$ discrete points indeed gives us the required discrete derivative of the function up to a factor $i$, and that the output vector is simply the average of the right-sided and left-sided discrete derivatives.