Physics 125a – Problem Set 5 – Due Nov 12, 2007

Version 1 – Nov 5, 2007

This problem set focuses on one-dimensional problems, Shankar Chapter 5 and Lecture Notes Section 5. Finally, some real quantum mechanics!

1. Find $\langle (\Delta X)^2 \rangle$ and $\langle (\Delta P)^2 \rangle$ for the eigenstates of the one-dimensional particle in a box. It is possible to do the calculation for all the odd or even modes at once. Hint: use a derivative to pull down factors of x, and also remember the normalization condition for the states.

2. Expanding boxes:

   (a) Shankar 5.2.1: A particle is in the ground state of a perfect box of length L. Suddenly, the box expands (symmetrically) to twice its size, leaving the wavefunction (initially) undisturbed. Show that the probability of finding the particle in the ground state of the new box is $\left(\frac{8}{3\pi}\right)^2$. (The key point in this problem is that *the wavefunction is initially undisturbed*. So, what is the initial state for the wavefunction in the new box?)

   (b) Shankar 5.2.4: Consider a particle of mass m in the state $|n\rangle$ of a perfect box of length L (*i.e.*, the state with $E = \frac{\hbar^2 \pi^2 n^2}{2 m L^2}$). Find the force $F = -\frac{\partial E}{\partial L}$ encountered when the walls are slowly pushed in, assuming the particle remains in the state $|n\rangle$ of the box as its size changes. Consider a classical particle of energy E_n in the box. Find its velocity, the frequency of collision on a given wall, the momentum transfer per collision, and hence the average force. Compare it to $-\frac{\partial E}{\partial L}$ computed above. Can you think of a criterion in the quantum mechanical case for "how slow is slow enough" for the rate at which the box walls are pushed in?

3. Shankar 5.2.2:

   (a) Show that for any normalized state $|\psi\rangle$, it holds that $\langle \psi | H | \psi \rangle \geq E_0$ where E_0 is the lowest-energy eigenvalue. (Hint: expand $|\psi\rangle$ in the eigenbasis of H.)

   (b) Prove the following theorem: Every attractive potential in one dimension has at least one bound state. Hint: Since V is attractive, if we define $V(\pm\infty) = 0$, it follows that $V(x) = -|V(x)|$ for all x. To show that there exists a bound state with $E < 0$, consider

$$\psi_\alpha(x) = \left(\frac{\alpha}{2\pi}\right)^{1/4} e^{-\frac{\alpha x^2}{4}} \tag{1}$$

   (Note that our α is defined slightly differently than in Shankar for consistency with how we write Gaussian wavefunctions in the Lecture Notes.) Calculate $E(\alpha) = \langle \psi_\alpha | H | \psi_\alpha \rangle$ for $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - |V(x)|$. Show that $E(\alpha)$ can be made negative by a suitable choice of α. The desired result follows from the application of the theorem proved above.

4. Bound states and scattering for a delta-function potential well.

   (a) Consider a potential of the form $V(x) = -a\, V_0\, \delta(x)$. Show that it admits a bound state of energy $E = -\frac{m\, a^2\, V_0^2}{2\, \hbar^2}$. Are there any other bound states?
   (b) Calculate the free states and reflection and transmission probabilities as functions of energy using the probability current.

   Hint: Solve the eigenvalue-eigenvector equation for the Hamiltonian outside the potential, requiring the appropriate asymptotic behavior and continuity at $x = 0$. Determine the change in the first derivative by calculating

$$\int_{-\epsilon}^{+\epsilon} dx\, \frac{d^2}{dx^2}\, \psi_{E,x}(x) \tag{2}$$

   (for $\epsilon \to 0$) using the eigenvalue-eigenvector equation.

5. Bound states and scattering for a finite square well. Consider a square well potential,

$$V(x) = \begin{cases} 0 & |x| \leq a \\ V_0 & |x| > a \end{cases} \tag{3}$$

   where $V_0 > 0$.

   (a) **Bound states**: Since when $V_0 \to \infty$, we have a box, let us guess what the lowering of the walls does to the states. First of all, all the bound states will have $E \leq V_0$. Second, the wave functions of the low-lying levels will look like those of the particle in a box, with the obvious difference that $\psi(x)$ will not vanish at the walls but instead spill out with an exponential tail. The eigenfunctions will still be even, odd, even, etc.
   Show that the even solutions have energies that satisfy the transcendental equation

$$k\, \tan(k\, a) = \kappa \tag{4}$$

   while the odd ones will have energies that satisfy

$$k\, \cot(k\, a) = -\kappa \tag{5}$$

   where k and $i\,\kappa$ are the real and complex wavenumbers inside and outside the well, respectively. Note that k and κ are related by

$$k^2 + \kappa^2 = \frac{2\, m\, V_0}{\hbar^2} \tag{6}$$

   Verify that as V_0 tends to ∞, we regain the levels of the particle in a box. The two transcendental equations 4 and 5 must be solved graphically (or numerically). In the $(\alpha = k\, a, \beta = \kappa\, a)$ plane, imagine a circle that obeys Equation 6. The bound states are then given by the intersections of the curve $\alpha \tan \alpha = \beta$ or $\alpha \cot \alpha = -\beta$ with the circle. (Remember α and β are positive.) Show that there is always one even solution and that there is no odd solution unless $V_0 \geq \frac{\hbar^2\, \pi^2}{2\, m\, L^2}$. What is E when V_0 just meets this requirement? Note that the general result from Problem 3 holds.
   (b) **Free states**: Now, consider free states associated with this potential. Find the eigenfunctions as a function of the energy E of the state for $E > V_0$. Calculate the reflection and transmission probabilities for right-going states incident on the well from the left side using the probability current. Plot the transmission probability as a function of E, making sure that you go to high enough energy that the wave in the well can complete at least 2 or 3 oscillations. What is the cause of the transmission resonances?