Physics 125a – Problem Set 5 – Due Nov 18, 2008

These problems cover Shankar 5.1-5.2 and Lecture Notes 5.1-5.3 on the free particle, particle in a box, and general considerations on bound and free states.

1. Consider the potential

\[ V(x) = \lim_{V_0 \to \infty} \begin{cases} V_0 & x < 0 \\ 0 & x \geq 0 \end{cases} \]  

What are the eigenstates of the Hamiltonian for a particle in this potential (including normalization)? Are the eigenstates of the Hamiltonian also eigenstates of the momentum operator? What is the classical explanation for your answer?

2. Find the expansion of the energy eigenstates \(|\psi_{E_n}\rangle\) for the particle in a box in terms of momentum eigenstates for the particle on the full real line,

\[ \langle x | p \rangle = \frac{1}{\sqrt{2\pi \hbar}} e^{i\frac{p x}{\hbar}} \]  

Plot \(P(p) = |\langle p | E_n \rangle|^2\) for \(n = 1, 2, 3\). Don’t forget to consider both positive and negative \(p\). In a qualitative way, how does the width of \(P(p)\) relate to the box size? For \(n > 1\), consider the widths of the positive and negative \(p\) components separately. Don’t explicitly calculate \(\langle (\Delta P)^2 \rangle\), it’s ugly; we just want you to see how the width scales with \(L\).

3. Suppose one has a problem in which the Hamiltonian suddenly changes at \(t = 0\); the classic example is the particle in a box with the box suddenly expanded in size at \(t = 0\). By integrating the Schrödinger Equation in time across the \(t = 0\) boundary, show that \(\psi_x(x, t)\) is continuous in \(t\); that is, the position-space wavefunction does not display any sudden changes at \(t = 0\). You should assume that the Hamiltonian is finite everywhere where the wavefunction is nonvanishing, both before and after the transition; we are assured of the former, but we must assume the latter.

4. Given a Hamiltonian with eigenstates \(|E_j\rangle\) ordered by the eigenvalues \(E_j\) with \(E_0 < E_1 < E_2 < \ldots\) (You may assume there are no degenerate eigenvalues.) Show that the element \(|\psi\rangle\) of the Hilbert space that minimizes \(\langle \psi | H | \psi \rangle\) must be the same state as the lowest-energy eigenstate of \(H\), \(|E_0\rangle\). Hint: the \(|E_j\rangle\) form a complete basis.

5. Find the Hamiltonian’s eigenstates for a particle in the potential

\[ V(x) = V_0 [\delta(x - a) + \delta(x + a)] \]  

Do this as follows. Solve the eigenvalue-eigenvector equation for this Hamiltonian independently in the three regions \(x < -a\), \(-a < x < a\), and \(x > a\) to obtain the form of the
wavefunction in the three regions separately, requiring the appropriate behavior at \( x = \pm \infty \). Obtain matching conditions between the three solutions at \( x = \pm a \) by techniques similar to what we used for the particle in a box and the step scattering potential to demonstrate that \( \psi_{E,x}(x) \) must be continuous at \( x = \pm a \) and to determine the change in the first derivative at \( x = \pm a \) by calculating

\[
\int_{\pm a-\epsilon}^{\pm a+\epsilon} dx \frac{d^2}{dx^2} \psi_{E,x}(x)
\]  

(4)

(for \( \epsilon \to 0 \)) using the eigenvalue-eigenvector equation.