Ay 123

Fall 2007

STELLAR STRUCTURE AND EVOLUTION

Final Exam: Solutions

1. Deriving the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\theta^n$$

from the equation of state

$$P = K\,\rho^\gamma$$

was done in class (see RSE notes §3.2).

The binding energy Ω can be derived as follows. Beginning with the equation of hydrostatic equilibrium:

$$\frac{dP}{dr} = \frac{-G\,M(r)}{r^2}\,\rho(r)$$

we multiply by $4\pi r^2$ and integrate from r=0 ⇒ R using $dM = 4\pi\,\rho\,r^2 dr$ to get

$$\int_0^R 4\pi\,r^3\,dP = -\int_0^R \frac{G\,M}{r}\rho\,4\pi\,r^2\,dr = -\int_0^M \frac{GM}{r}\,dM$$

where the last term is Ω. Integrate the first term by parts:

$$\int_0^R 4\pi\,r^3\,dP = 4\pi\,Pr^3\Big|_0^R - 4\pi\int_0^R 3r^2\,P\,dr = -3\int_0^R P\,4\pi\,r^2 dr$$

where we have assumed $P(R) \simeq 0$. So, finally

$$-\int_0^R P\,4\pi\,r^2 dr = \frac{\Omega}{3} \quad [1]$$

We can also write

$$\Omega = -\int_0^M \frac{GM}{r} dM$$

and integrating this by parts ($u = 1/r, dv = M\,dM$) gives

$$\Omega = -\frac{GM}{2R} + \frac{M^2}{2r}\Big|_{M=0} - \int_0^R \frac{GM^2}{2r^2} dr$$

The second term vanishes assuming ρ is finite at $M = r = 0$. Using hydrostatic equilibriium, we can rewrite this as:

$$-\frac{GM}{r^2} = \frac{1}{\rho}\frac{dP}{dr} \rightarrow \Omega = -\frac{GM^2}{2R} + \frac{1}{2}\int_0^R \frac{M}{\rho}\frac{dP}{dr}dr \quad [2]$$

Integrate this by parts ($u = M/\rho, dv = dP/dr$) gives:

$$\Omega = -\frac{GM^2}{2R} - \frac{1}{2}\int_0^R \frac{P}{\rho}\frac{dM}{dr}dr + \frac{1}{2}\int_0^R \frac{P\,M}{\rho^2}\frac{d\rho}{dr}dr \quad [3]$$

again, assuming $P(R) \simeq 0$.

Now since $dM = 4\pi\, r^2\, \rho\, dr$, the second term on the RHS in [3] above simplifies using [1] to

$$-\frac{1}{2}\int_0^R P\, 4\pi\, r^2\, dr = \frac{1}{6}\Omega \quad [4]$$

And the third term in [3] can be simplified using the polytropic equation of state $P = K\,\rho^{(n+1)/n}$ to become

$$dP = \frac{n+1}{n}K\,\rho^{1/n}\,d\rho \Rightarrow \frac{dP}{P} = \frac{n+1}{n}\frac{d\rho}{\rho} \Rightarrow \frac{1}{\rho}\frac{d\rho}{dr} = \frac{n}{n+1}\frac{1}{P}\frac{dP}{dr} \quad [5]$$

Combining [3],[4] and [5], we have:

$$\frac{5}{6}\Omega = -\frac{GM^2}{2R} + \frac{n}{2(n+1)}\int_0^R \frac{M}{\rho}\frac{dP}{dr}\,dr$$

However, from [2] we can see the last term can be rewritten in terms of Ω, M and R to yield

$$\frac{5}{6}\Omega = -\frac{GM^2}{2R} + \frac{n}{(n+1)}(\Omega + \frac{GM^2}{2R})$$

which simplifies to

$$\Omega = -\frac{3}{(5-n)}\frac{GM^2}{R}$$

Examination of this expression shows that the binding energy becomes positive for $n > 5$ so the star is not gravitationally bound. A stable solution can only be achieved for $n \leq 5$.

Polytropes can be useful in the following stages of stellar evolution. n=1.5 (protostars, non-relativistic degenerate stars), n=3 (Eddington's solution - part radiation pressure, part gas pressure -, relativistic degenerate gas), $n = \infty$ (relativistic neutron star).

2. Much of this was covered in class (see RSE notes §6.1). The general pressure integral is

$$P = \frac{1}{3}\int_0^{p_F} v\,p\,f(p)dp$$

where $f(p)$ is the space density of electrons with momenta $p \rightarrow p + dp$.

The phase space density at T=0 is $2/h^3$ for $p < p_F$, otherwise 0, so

$$f(p)dp = \frac{2}{h^3} 4\pi p^2 dp \quad (p < p_F)$$

so

$$P = \frac{8\pi}{3h^3} \int_0^{p_F} v p^3 dp$$

Now

$$v = \frac{dE}{dp} = \frac{d}{dp}\sqrt{(mc^2)^2 + (pc)^2} = c\,\frac{p/(mc)}{\sqrt{1 + (p/mc)^2}}$$

hence

$$P = \frac{8\pi c}{3h^3} \int_0^{p_F} \frac{p/mc}{\sqrt{1 + (p/mc)^2}} p^3 dp$$

In the *non-relativistic case*: $p << mc$ and $\sqrt{1 + (p/mc)^2} \simeq 1$ hence

$$P = \frac{8\pi c}{3h^3} \frac{1}{mc} \int_0^{p_F} p^4 dp = \frac{8\pi}{15mh^3} p_F^5$$

In the *relativistic case*: $p >> mc$ and $\sqrt{1 + (p/mc)^2} \simeq p/mc$ hence

$$P = \frac{8\pi c}{3h^3} \frac{1}{mc} \int_0^{p_F} p^3 dp = \frac{2\pi c}{3h^3} p_F^4$$

In these expressions, p_F is given in terms of the electron density as

$$n = \frac{2}{h^3} \int_0^{p_F} 4\pi p^2 dp = \frac{8\pi}{3h^3} p_F^3$$

The transition occurs when $p \simeq mc$ corresponding to a number density n

$$n = \frac{8\pi}{3h^3} (mc)^3 = \frac{\rho}{m_H}$$

where ρ is the sought-after critical density, whence

$$\rho_{crit} = \frac{8\pi}{3} (\frac{mc}{h})^3 m_H$$

In the non-relativistic case, we note that $P \propto p_F^5 \propto n^{\frac{5}{3}} \propto \frac{M^{5/3}}{R^5}$ while the pressure needed to maintain hydrostatic equilibrium $\propto \frac{M^2}{R^4}$. So for an equilibrium configuration $R \propto M^{-\frac{1}{3}}$ and so as M increases, R decreases (see RSE notes §6.2)

In the relativistic limit $P \propto p_F^4 \propto n^{\frac{4}{3}} \propto \frac{M^{4/3}}{R^4}$ which has the same R-dependence as the gravitational force per unit volume. The degeneracy pressure can thus no longer support the star beyond a certain critical mass (see RSE notes §6.3).

3. My table of modes of energy transfer looks like the following:

	Center	Half-mass radius	Photosphere
$0.1M_\odot$	Convection	Convection	Convection
$1M_\odot$	Radiation	Radiation	Convection
$2M_\odot$	Convection	Radiation	Radiation
White Dwarf	Conduction	Conduction	Radiation

For the problem, we begin with the diffusion equation

$$\frac{d}{dr}(\frac{1}{3}aT^4) = -\frac{F\kappa\rho}{c}$$

into which inserting hydrostatic equilibrium yields

$$\frac{4}{3}aT^3\frac{dT}{dr} = \frac{F\kappa}{c}\frac{r^2}{GM}\frac{dP}{dr}$$

hence

$$\frac{dT}{dP} = \frac{3}{4}\frac{1}{aT^3}\frac{F\kappa}{c}\frac{r^2}{GM}$$

But $F = \sigma T_{eff}^4$ and $a = 4\sigma/c$, so inserting the formula gives:

$$\frac{dT}{dP} = \frac{3}{16}T_{eff}^4\kappa_0 P^{\alpha-1}T^{1-\beta}\frac{1}{GM}$$

Assuming T=0 when P=0 gives a solution:

$$T^\beta = \frac{3}{16}T_{eff}^4\frac{\kappa_0}{8}\frac{\beta}{\alpha}P^\alpha$$

Rewriting the differential equation as:

$$\frac{P}{T}\frac{dT}{dP} = \frac{3}{16}T_{eff}^4\frac{\kappa_0}{8}P^\alpha T^{-\beta}$$

or

$$\frac{dlogT}{dlogP} = \frac{3}{16}T_{eff}^4\frac{\kappa_0}{8}P^\alpha T^{-\beta} = \frac{\alpha}{\beta}$$

The condition for convective stability is that this gradient exceed $\Gamma_{ad} = \frac{2}{5}$, i.e.

$$\alpha > \frac{2}{5}\beta$$

.

4. The note in the class web site on radiative transfer covers most of this question. In particular, the formal solution for outward flowing rays with $\mu > 0$, given there is

$$I(\tau, \mu, \nu) \; = \; \int_{\tau}^{\infty} S_{\nu}(t) exp[-(t-\tau)/\mu] \frac{dt}{\mu}.$$

This, when integrated over a source function linear in optical depth, yields

$$I(\tau, \mu, \nu) \; = \; a_{\nu} + b\nu(\tau + \mu).$$

5. Parts (a) and (b) of this problem are covered in the notes on the class web site about absorption lines.

c) What I was after towards the end of this part is that the mean depth of formation of this line is gradually pushed toward the surface of the Sun as the abundance of Ca is increased. If the abundance becomes high enough, then the center of the line will eventually become formed in the chromosphere of the Sun, above the temperature minimum. The line will appear to have a small emission core. (This is in fact observed in the Sun and in other late-type stars, and is useful as a diagnostic of stellar activity.)

Even in a classical LTE stellar atmosphere, which does not have a chromosphere or corona, and where $T(\tau = 0)$ is the minimum in the atmosphere, there is still a minimum for the residual intensity of any very strong line, corresponding to $B_{\nu}[T(\tau = 0)]/B_{\nu}[T(\tau_{cont} = 2/3)]$.

6. a) We need to look at the ratio $\kappa(line)/\kappa(cont)$ evaluated at ν corresponding to the line center. For a cool star, $\kappa(cont)$ is that of $H^- \; \propto \; T^{-2.5} \; P_e \; e^{0.75/kT}$, which results from the Saha equation applied to the system H^- ionizing to neutral hydrogen, with an ionization potential of 0.75 eV. Balmer lines arise from the $n = 2$ level of neutral hydrogen, while hydrogen in the atmosphere of a cool star is mostly neutral, so $\kappa(line) \; \propto \; e^{-\chi/kT}$, where χ is the excitation for the appropriate level, the $n = 2$ level, which has $\chi \; = \; 0.75 \times 13.6 \; = \; 10.2$ eV. The Balmer lines are very strong, so the line profiles are dominated by damping wings, and we must multiply the level population factor $e^{-\chi/kT}$ by P to take into account the damping.

This factor cancels the P_e in the denominator, so the line strength is approximately independent of pressure (i.e. of surface gravity), but increases strongly with temperature as the $n = 2$ level, with its high excitation potential, is increasingly populated.

For hot stars, the continuum opacity is primarily bound-free absorption of H which at the frequency of Hγ arises from the $n = 3$ level. So now there is nothing to cancel the P from the damping term, and there is a strong surface gravity dependence, with Hγ being broader (stronger) for larger P, i.e. higher surface gravity. The temperature dependence arises from populating the 3rd level for the continuous opacity versus the 2nd level for the line opacity, and from the increasing ionization of H at higher temperatures.

b) For fully ionized H, i.e. in the center of a low mass star which has just arrived on the main sequence, $n = n(H) + n(e)$, hence $P(gas) = 2P_e$. For fully ionized He, which condition prevails in the non-degenerate center of a 2 $M_\odot$ red giant, there are 2 electrons for every He nucleus, and there is no H, it all having been burned to He. So $P(gas) = 1.5P_e$. The stellar atmosphere of a hot star corresponds to fully ionized H, while in the atmosphere of a cool star, H is mostly neutral and $P_e << P(gas)$.

There is no metallicity dependence when the gas is fully ionized and consists largely of H and/or He. But for a cool gas, where H and He are almost entirely neutral, metals, whose first ionization potential is often ~ 5 eV (less than half that of H), contribute a non-trivial fraction of the electrons. Thus a higher metallicity gas will have a somewhat larger P_e for a given temperature and total number density (dominated by H and/or He).

7. After H is burned to He via the $p-p$ chain or the CNO cycle, the triple-α process burns He to ^{12}C. Then ^{12}C $+\alpha$ yields ^{16}O. Another α capture produces ^{20}Ne from ^{16}O.

The rest of this question is similar to problem 3 in homework set 7. One must also add in the details for calculating the chemical evolution. These include, among other factors, the yield of O produced by nuclear burning in a star as a function of its mass and the distance from the center of the star at the end of nuclear burning phases in the star, the amount of O ejected into the ISM by winds or by a SN as a function of mass, the lifetime of the star and the timescale for mass ejection.

8. Most of this question is classwork (see RSE notes §7.2). The two primary differences between neutron and electron degeneracy are: (i) relativistic neutrons can have two comparable components of energy density - a rest mass component and a kinetic energy component, whereas the former dominates for electrons, (ii) when neutrons are closely packed, the strong force can be significant, affecting the equation of state and making it 'non-ideal'. Recent modeling of these effects suggests the maximum mass of a neutron star is around 2 $M_\odot$. In class we derived the nuclear density of a core of Fe^{56} to be around 8×10^{14} gm cm^{-3} whence the radius of a 1.4 $M_\odot$ neutron star is about 15 kms. The total energy released in the collapse to a core of this radius is about 2×10^{52} ergs (see RSE notes §7.3.2).

Type Ia supernovae have been more effective in cosmology than Type II because they are much more luminous (so can be seen to redshifts beyond unity) and because they can, via their light curves, be calibrated as fairly homogeneous 'standard candles' ($\sigma_m \simeq$0.3 mag). But this is at the expense of not knowing precisely how they explode or whether there are environmental or evolutionary variations in their properties.

TMT offers the real prospect of using the more well-understood Type IIP's (plateau) events for cosmology. The basic method we would need to use is a variant of the *Baade-Wesselink* method discussed in class (RSE notes §7.7) which requires the plateau brightness (photometry), the expansion velocity (spectrum) and the redshift. This gives the angular diameter distance versus redshift.