Ay123

Fall 2007

STELLAR STRUCTURE AND EVOLUTION

FINAL EXAM

Due Friday, December 14th 2007 at Noon

This is a **closed book exam**. No notes (not even your lecture notes) or consultation. An unprogrammeable handheld calculator is acceptable, if you need it. Please take three straight hours in a quiet, isolated, location and do not look at the problems until you are ready to begin. All questions carry equal weight.

Some possibly useful constants are provided at the end.

Please hand in your completed paper to Judy McClain **in person** by Friday December 14th at Noon.

Attempt FIVE questions

1. For an assumed equation of state, a solution for the run of pressure, temperature and density can be obtained for a star in hydrostatic equilibrium. Show that, for a polytropic equation of state:

$$P = K\,\rho^{\gamma}$$

where K and γ are independent of radius, the density dependence is given by the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\,(\xi^2\,\frac{d\theta}{d\xi}) = -\theta^n$$

where the density and radius are expressed in terms of dimensionless variables via $\rho = \lambda\,\theta^n$, the radius $r = \alpha\,\xi$, respectively and the index $n = (\gamma - 1)^{-1}$.

Accordingly, show that the gravitational potential energy of a star is then given by the expression

$$\Omega = -\frac{3\,G\,M^2}{(5\,-\,n)R}$$

What is the implication of this result for the range of n applicable to stable stars. Summarize the situations during stellar evolution for which polytropic solutions are appropriate, explaining clearly the value of n that is applicable in each case.

2. Show that, for the general case, the pressure in an electron gas is given by the integral of the kind

$$P = \frac{1}{3}\int_0^{P_F} v\,p\,f(p)\,dp = \frac{8\pi\,c}{3h^3}\int_0^{p_F} \frac{p\,/\,(m_e\,c)}{\sqrt{1\,+\,p^2\,/\,(m_e^2\,c^2)}}\,p^3\,dp$$

where m_e is the electron rest mass and p_F the Fermi momentum.

Now consider how this expression can be simplified in the non-relativistic and ultra-relativistic case. Derive the critical density, ρ_{crit}, for the transition between these two extremes assuming, for simplicity, that $n_e = \rho\,/m_H$ in determining the electron number density.

Explain, in simple terms (i.e. without further lengthy derivations), why the radius of a degenerate non-relativistic white dwarf becomes smaller as its mass increases and why, in the ultrarelativistic case, there is a maximum stable mass.

3. Make a 4 × 3 table which names the major energy transport mechanisms (i) close to the center, (ii) at the radius containing half the mass, and (iii) just below the photosphere. Do this for main sequence stars of mass: (a) 0.1 $M_\odot$, (b) 1 $M_\odot$, (c) 2 $M_\odot$ and (d) a white dwarf of 0.5 $M_\odot$.

In the outer layers of a star, assume $L(r) = L$ and $M(r) = M$, i.e. constants. Assume the opacity is a power law in pressure and temperature, viz:

$$\kappa = \kappa_0\,P^{\alpha-1}\,T^{4-\beta}$$

where κ_0, α and β are constants. Under these conditions, derive a differential equation and boundary condition that will serve to determine $T(P)$ in the outer layers. Investigate the behavior of $T(P)$ at great depth and show that the outer layers must be convective if

$$\alpha > \frac{2}{5}\,\beta > 0$$

taking the adiabatic gradient to be $\frac{2}{5}$.

4. (The Eddington Approximation) Assume a source function increasing linearly with optical depth, i.e.

$$S_\nu(\tau_\nu) = a_\nu + b_\nu\tau_\nu.$$

(a) Define optical depth τ, specific intensity I_ν and the source function S_ν for a radiation field in a gas. Explain what is meant by the Rosseland mean opacity and how it is used. Give dimensions for each of these variables.

(b) Integrate the equation of radiative transfer to find $I_\nu(\overline{\tau})$, where $\overline{\tau}$ is an appropriate mean over frequency of τ_ν.

(c) Show that the effective temperature T_{eff} is equal to $T(\overline{\tau} = 2/3)$.

(d) For the same source function, derive an expression for the second moment of the radiation field, $K_\nu = 0.5 \int_{-1}^{+1} I_\nu \mu^2 d\mu$, where $\mu = cos(\theta)$.

(e) What does your result from part (c) imply about the surface radiation pressure from photons of frequency ν, in particular its relationship to the source function at some characteristic optical depth? What is this characteristic optical depth?

5. (a) Give the equation for the absorption coefficient per atom of calcium as a function of frequency for a line with line center ν_0 from a level with excitation potential χ of neutral Ca. Define each symbol and indicate the physical effects contributing. (Extra credit if you remember the constant factor correctly in terms of π, the charge of the electron, etc.)

(b) Define the equivalent width W_ν of an absorption line. Sketch the curve of growth for the CaI line of part (a) for a classical LTE model atmosphere, label the axes, and indicate what causes the different regimes of the relationship between abundance of the relevant atom and W_ν. Which physical effects determine the width of the line (i.e. full width half maximum, for example) ? Which determine the central depth of a line ?

(c) Consider this CaI line in the spectrum of the Sun. Assume it is a resonance line, arising from the ground state of Ca. Sketch its central depth, $F(line, \nu = \nu_0)/F(\text{continuum}, \nu = \nu_0)$, as a function of Ca abundance assuming that the ratio n(Ca)/n(H) ranges from 10^{-2} of the solar ratio to 10^3 times that of the sun. Discuss why your curve behaves as sketched. What is the minimum in the center of the line for an extremely strong absorption line of neutral Ca and why does a minimum occur ?

6. (a) At spectral type B3 ($T_{eff} \sim 15,000$ K) the Hγ absorption line is approximately 3 times stronger in a giant than in a supergiant. Yet in G type stars ($T_{eff} \sim 5500$ K), Hγ shows little gravity sensitivity, but depends sensitively on temperature. Discuss the continuous opacity, the line opacity, and the line broadening mechanism in these two cases, and use your results to explain the behavior of Hγ described above.

(b) Give a relationshiop between the electron pressure P_e and the gas pressure P_g in the center of a low mass star which has just arrived on the main sequence, in the (non-degenerate) center of a 2 $M_\odot$ red giant, in the stellar atmosphere of a hot star ($T_{eff} \sim 15,000$ K), and in the stellar atmosphere of a cool star ($T_{eff} \sim 4,000$ K). What is the dependence on metallicity in each of these cases and why ?

7. (a) Oxygen is the third most abundant element in the Universe, after H and He. Describe the main chain of nuclear reactions which occur in stars to produce ^{16}O beginning with hydrogen, and to burn ^{16}O as well. Indicate the required temperature for each group of reactions to proceed, and suggest in what mass range of stars each might occur.

(b) Indicate how to set up, but do not attempt to solve, a calculation to predict the O/H ratio in the interstellar gas in the disk of our galaxy as a function of time since the formation of the galaxy. Ignore accretion of material into the disk after its initial formation. Ignore radial flows within the galaxy. Discuss the factors and relationships that would be required as input to solve this problem.

8. State the two physical differences between the behavior of a degenerate neutron gas and that of a degenerate electron gas. How do these affect the upper mass limit that we can expect for a neutron star? Estimate the radius of a neutron star of mass 1.4 $M_\odot$.

Estimate the total energy than can power a supernova if a collapsing core of 1.4 solar masses of pure Fe^{56} shrinks to the size of a neutron star.

Why have supernova of Type Ia been so much effective in cosmological distance measurements than Type II supernova given we have a much better physical understanding of the latter events? Explain how, with the increased aperture of TMT, it might be possible to confirm the cosmic acceleration with Type II supernovae? What measurements would need to be made?

Proton mass: $m_P = 1.67262 \times 10^{-24}$ g
Electron mass: $m_e = 9.11 \times 10^{-28}$ g
Neutron mass: $M_N = 1.67493 \times 10^{-24}$ g
Radius of Fe^{56} nucleus $\simeq 3.\ 10^{-13}$ cm
Planck's constant: $h = 6.6 \times 10^{-27}$ ergs s
Solar mass: $M_\odot = 1.99 \times 10^{33}$ g
Solar luminosity: $L_\odot = 3.85 \times 10^{33}$ ergs sec^{-1}
Solar radius: $R_\odot = 6.96 \times 10^{10}$ cm
Boltzmann's constant: $k = 1.38 \times 10^{-16}$ erg K^{-1}
Stefan-Boltzmann constant: $\sigma = 5.67 \times 10^{-5}$ ergs cm^{-2} K^{-4} s^{-1}
Energy conversion: 1 eV = 1.6 $\times 10^{-12}$ ergs