1. Homology Relations for Main Sequence Stars for Ay 153

Our goal is to be able to use 1 stellar model as an initial model and scale it to a family of solutions for closely related models. HKT discuss this in section 1.6, calling it stellar dimensional analysis.

Consider $M(r)$, the enclosed mass within a radius r for a spherically symmetric star, as the fundamental variable. We blow up or contract the star in radius to get different total masses. We denote the original model with subscript "*"; it has a total mass of M_*. We are in effect assuming that $\rho(r)/ <\rho>$ is the same function independent of total mass. The figure illustrates the radii of constant enclosed mass as a fraction of total stellar mass in 20% increments for a pair of homologous stars and for a third star which is not homologous to the other two.

For homology, $r_2(M) = r_*(M)\ (M_2/M_*)^a$, where a is some real number. If we stick to a regime of stellar total mass such that M_2/M_* close enough to unity, then we will assume homology prevails, i.e. the stellar model for M_2 has the same general structure as the initial one and a is a constant.

Within that small mass range, for a new model of mass M, we introduce powers $a,\ b,\ c,\ d$ and e which represent the behvaior of each of the key variables, and we assume:

$$\mathcal{M} = M/M_*$$

$$r = (M/M_*)^a\ \overline{r}(\mathcal{M})$$

$$\rho = (M/M_*)^b\ \overline{\rho}(\mathcal{M})$$

$$L = (M/M_*)^c\ \overline{L}(\mathcal{M})$$

$$T = (M/M_*)^d\ \overline{T}(\mathcal{M})$$

$$P = (M/M_*)^e\ \overline{P}(\mathcal{M})$$

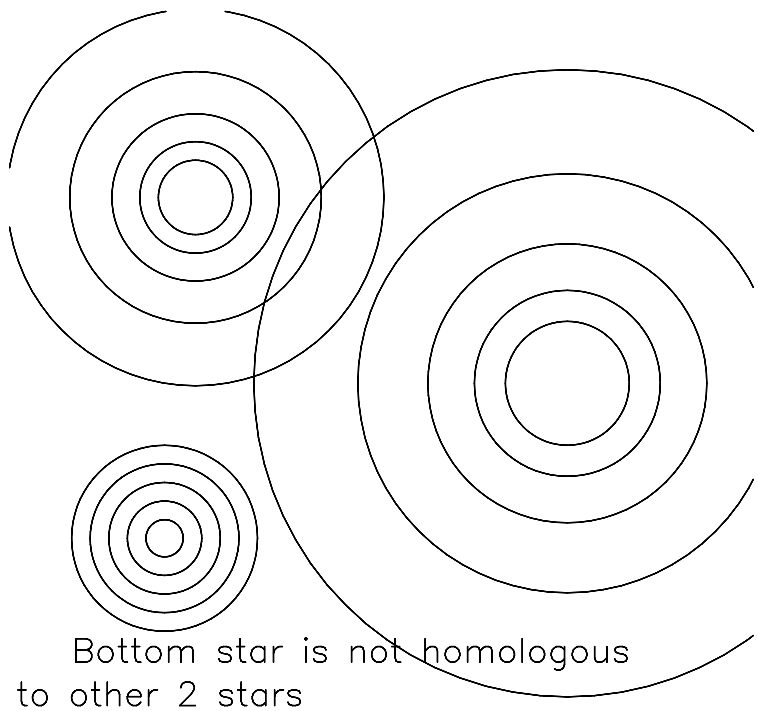

Fig. 1.— Two homologous stars and one not homologous to the other two. Radii in 20% increments of the total mass are shown for each star.

1.1. Solving for the Exponents

We apply this concept to the family of main sequence stars. We assume radiative energy transport, $P = P(gas)$ only, or $P = P(rad)$ only, here we use $P(gas)$.

We parameterize the nuclear energy generation rate ϵ as $\epsilon_0 \rho T^\eta$ and the opacity as $\kappa_0 \rho^{\lambda-1}/T^{\nu-3}$.

We now use the equations of stellar structure to solve for the exponents $a,\ b,\ c,\ d$ and e in terms of $\eta,\ \lambda,\ \nu$ and constants.

$$\rho(M) = [\frac{M(r)}{M_0(r)}]^b \ \rho(M_*)$$

$$\frac{dM}{dr} = 4\pi r^2 \rho, \qquad \frac{dr}{dM} = \frac{1}{4\pi r^2 \rho}$$

$$M_*^a \ \frac{d\overline{r}}{dM} = [4\pi \overline{r}^2 \ M_*^{2a} \ \overline{\rho}(M) \ M_*^b]^{-1}$$

Collecting the powers of M_* we get:

$$M_*^{a-1} \ \frac{d\overline{r}}{dM_*} = [M_*^{2a+b} \ 4\pi \overline{r}^2 \ \ \overline{\rho}(M)]^{-1}$$

So we have our first equation relating the power law exponents:

$$3a \ + b - 1 = 0, \qquad \text{and} \qquad \frac{dM_*}{d\overline{r}} = 4\pi \overline{r}^2 \overline{\rho}$$

From the gas law, $P = \rho k T/(\mu m_H)$, we get

$$e \;=\; b \;+\; d.$$

If the star is very hot, then $P(rad)$ dominates, and we get $e \;=\; 4d$ instead.

From the luminosity eqution we get:

$$\frac{dL}{dM} \;=\; \epsilon \;=\; \epsilon_0 \;\rho\; T^\eta \;=\; \epsilon_0 \;\overline{\rho}\; \mathcal{M}_*^b \; \mathcal{M}_*^{\eta d}\; \overline{T}^\eta$$

Collecting the powers, we get another equation for the exponents,

$$c \;-\; 1 \;=\; b \;+\; \eta d$$

We get two more equations relating the power law exponents from hydrostatic equilibrium and for radiative transfer of energy. They are:

$$\frac{dP}{dM} \;=\; -\frac{\rho G M}{r^2(4\pi r^2 \rho)} \;=\; -\frac{GM}{4\pi r^4} \quad\rightarrow\quad 4a \;+e \;=\; 2$$

$$\frac{dT}{dM} \;=\; -\frac{3\kappa L}{64\pi^2 a c r^4 T^3} \;=\; -\frac{3\kappa_0 \rho^{\lambda-1} L}{64\pi^2 a c r^4 T^\nu}$$

Collecting the powers, we get another equation for the exponents,

$$b(\lambda-1)+c+1 \;=\; 4a+d(\nu-1)$$

We now have 5 linear equations in the 5 unknown powers $a,\ b,\ c,\ d$ and e, which can be solved once κ and ϵ are specified.

We consider 2 typical cases. For low mass stars $1 < M < 3\ M_\odot$ we adopt ϵ appropriate to the $p - p$ chain and Kramer's opacity. For hotter stars, we adopt Thomson scattering for the opacity and ϵ for the CN cycle. Below $1 M_\odot$ the assumption that convection is not important breaks down as stars become fully convective; the formulae given above then need to be modified. For the most massive stars they also need modification as P_{rad} needs to be included.

We then have the following solutions for the exponents listed in the Table below.

Table 1. Solutions for Homology Exponents

Case	ν	λ	η	a	b	c	d	e
Low mass	13/2	2	4	1/13	10/13	71/13	12/13	22/13
High mass	3	1	16	15/19	−26/19	3	4/19	−22/19

We can now derive relations such as

$$L = (\frac{M}{\overline{\overline{M}}})^c \ \overline{L}(\overline{M}) \qquad L \propto M^c \qquad L \propto M^{5.5} \text{ Low mass, } \propto M^3 \text{ High mass}$$

$$R = (\frac{M}{\overline{\overline{M}}})^a \ \overline{R}(\overline{M}) \qquad R \propto M^a \qquad \propto M^{0.1} \text{ Low mass, } \propto M^{0.8} \text{ High mass}$$

$$\rho_c = (\frac{M}{\overline{\overline{M}}})^b \ \overline{\rho}\overline{M}) \qquad \rho_c \propto M^b \qquad \rho_c \propto M^{0.8} \text{ Low mass, } \propto M^{-1.3} \text{ High mass}$$

A check of the above formulae: We know $\rho_c \propto M/R^3$. For high mass stars, we find from the value given in the table for the exponent a that $\rho_c \propto M/M^{2.4} \propto M^{-1.4}$. The solution for b given in the table is -1.36, equal to -1.4 to within the round off error. Note that higher mass main sequence stars have lower ρ_c.

For T and P we find:

$$T_c = (\frac{M}{\overline{\overline{M}}})^d \ \overline{T_c}(\overline{M}) \qquad T_c \propto M^{0.9} \text{ Low mass, } \propto M^{0.2} \text{ High mass}$$

$$P_c = (\frac{M}{\overline{\overline{M}}})^e \ \overline{P_c}(\overline{M}) \qquad P_c \propto M^{1.7} \text{ Low mass, } \propto M^{-1.2} \text{ High mass}$$

Note the modest dependence of T_c on mass, especially for high mass stars.

Another check: The perfect gas law implies $P_c \propto \rho_c T_c$. Using values for b and d from the table for high mass stars we get $P_c \propto M^{-1.3} \ M^{0.2} \propto M^{-1.1}$. The value of the exponent e for pressure is given as -1.15, in good agreement with the above.

To complete the characterization of the main sequence we give forms for T_{eff} and then L as a function of T_{eff}.

$$T_{eff} \propto (L/R^2)^{1/4} \propto M^{(c-2a)/4} \propto M^{1.3} \text{ Low mass}, \propto M^{0.4} \text{ High mass}$$

We get $L \propto T_{eff}^{4.2}$ for low mass stars and to $T_{eff}^{7.5}$ for high mass stars. A schematic HR diagram for main sequence stars would then be as indicated in the sketch below.

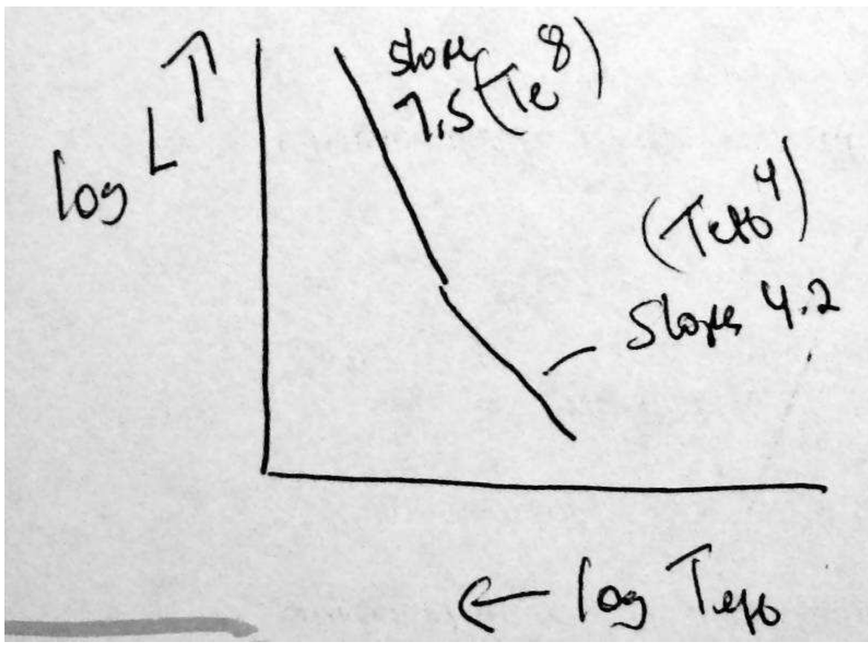

Fig. 2.— A sketch of the HR diagram ($\log(T_{eff}$ vs. $\log(L)$) for main sequence stars with the power law slopes derived above via the homology relations.

1.2. Applications of Homology Relations

We can use these results to calculate main sequence lifetimes, with timescale $\tau \propto M/L$. For for high mass stars using the relations derived above we find $\tau \propto M^{-2}$.

We can estimate the lowest possible mass for main sequence stars by setting T_C to the minimum temperature for the $p - p$ chain to produce ^{4}He. Then

$$[\frac{M_{low}}{M_\odot}]^d = T_c(min)/T_c(Sun).$$

We adopt $d = 0.9$ (the value for low mass stars given in the table) and $T_{min} = 4 \times 10^6$ K. This gives 0.01 $M_\odot$ as the lower mass limit for main sequence stars, not a bad estimate.

NOTE: KW gives a slightly more complicated treatment of this subject, introducing extra parameters, for example, $\rho \propto P^\alpha \, T^{-\delta} \, \mu^\phi$