

1. Numerical Methods to Compute Stellar Models

Problem – boundary conditions are split between the center of the star ($r = 0$, $M(r) = 0$) and surface, where $r = R$ (the radius of the star is not known initially, only the mass and chemical composition) and P and T are small.

WARNING – this document is incomplete, only the sections to be used later are given here.

1.1. Analytical Solutions

Analytical solutions to the equations of stellar structure are useful as educational tools and as a means of gaining insight into the effect of varying the stellar mass and other parameters. They are also useful as initial guesses for exact solutions which must be numerically calculated with an iterative scheme.

Polytropes where $P \propto \rho^n$ can be analytically solved for some values of n . When an analytical solution is not possible, polytropes are straightforward to integrate numerically without any iteration being required.

Another case which is straightforward to solve occurs if one assumes a density linearly related to radius, i.e., the linear density law, $\rho = \rho_c(1 - r/R)$. But this not a very realistic case.

A more realistic assumption for $P(r)$ can be obtained by finding a functional form that matches the desired derivative dP/dr near the center of the star and near its surface. Near the center

$$\frac{dP}{dr} = - \frac{4\pi}{3} G \rho_c^2 r \quad \text{since } M(r) = \frac{4\pi}{3} \rho_c r^3 \quad \text{for } r \approx 0$$

Near the surface, on the other hand,

$$\frac{dP}{dr} = - \rho \frac{GM^*}{R_*^2} \approx 0.$$

A guess of a functional form for $P(r)$ which has these properties for dP/dr , is smooth, monotonically decreases as r increases, is simple and integratable was given by Clayton, and is called Clayton's pressure function,

$$\frac{dP}{dr} = - \frac{4\pi G \rho_c^2 r}{3} e^{-r^2/\alpha^2}$$

This function has one free parameter, the scale length α . It has the desired behavior for $r \sim 0$ and, if α/R small, also for $r \sim R$. It can be integrated analytically to get (setting $P = 0$ at $r = R$):

$$P(r) = \frac{2\pi}{3} G \rho_c^2 \alpha^2 [e^{-r^2/\alpha^2} - e^{-R^2/\alpha^2}]$$

One can then integrate $GM(r)dM = -4\pi r^4 dP$ (the combined hydrostatic equilibrium plus mass conservation equations) to get $M(r)$, i.e.

$$M(r) = \frac{4\pi}{3} \alpha^3 \rho_c \phi(x)$$

where $x = r/\alpha$ and

$$\phi(x) = \int_0^x (x')^5 e^{-x'^2} dx'$$

The definite integral $\phi(x)$ has a (rather messy) analytical form, see, e.g. Phillips, eq. 5-27.

One can then find

$$\rho(r) = \frac{1}{4\pi r^2} \frac{dM}{dr} = \rho_c \left[\frac{x^3 e^{-x^2}}{\phi(x)} \right]$$

The equation of state can be used to obtain $T(r)$ once $P(r)$ and $\rho(r)$ are known.

For small r , one finds

$$\rho(r) = \rho_c \left[1 - \frac{5}{8} \frac{r^2}{\alpha^2} \right] \quad \text{and} \quad T(r) = T_c \left[1 - \frac{3}{8} \frac{r^2}{\alpha^2} \right]$$

(plus higher order terms).

For stars with high central density concentration ($\rho_c / \langle \rho \rangle$ large), such as the Sun, $\alpha \ll R$, $\exp(-\alpha^2/R^2)$ is small, and the above expressions simplify. A useful expression for the central pressure, which we will use again later, is

$$P_c \approx \left[\frac{\pi}{36} \right]^{1/3} G M^{2/3} \rho_c^{4/3} = 0.44 G M^{2/3} \rho_c^{4/3}.$$

This is close to the result for a $n = 1.5$ polytrope for which the numerical factor in the equation for P_c is 0.48, instead of the 0.44 obtained above.

1.2. Surface Boundary Conditions

Better surface boundary conditions than P and $T \approx 0$ can be derived from model atmospheres, as we will do later. The hydrostatic equilibrium equation in a stellar atmosphere simplifies to (approximately)

$$\frac{dP}{d\tau} = \frac{g}{\kappa}$$

where τ is the optical depth and κ is the absorption coefficient. Choosing a suitable optical depth (here $\tau = 3/2$), and assuming constant g and κ , one gets $P = (3/2)g\kappa$. With approximations to be discussed in the lecture on stellar atmosphere we also can get an expression for $T(\tau)$,

$$T^4 = \frac{3}{4} T_{eff}^4 (\tau + 2/3)$$

We again choose $\tau = 3/2$ to define the surface boundary conditions for a calculation of a full stellar model.

It should be noted that in stellar atmosphere calculations, the independent variable should be τ or P rather than $M(r)$ or r as the enclosed mass and radius both change little over the small radial extent of the atmosphere for stars.

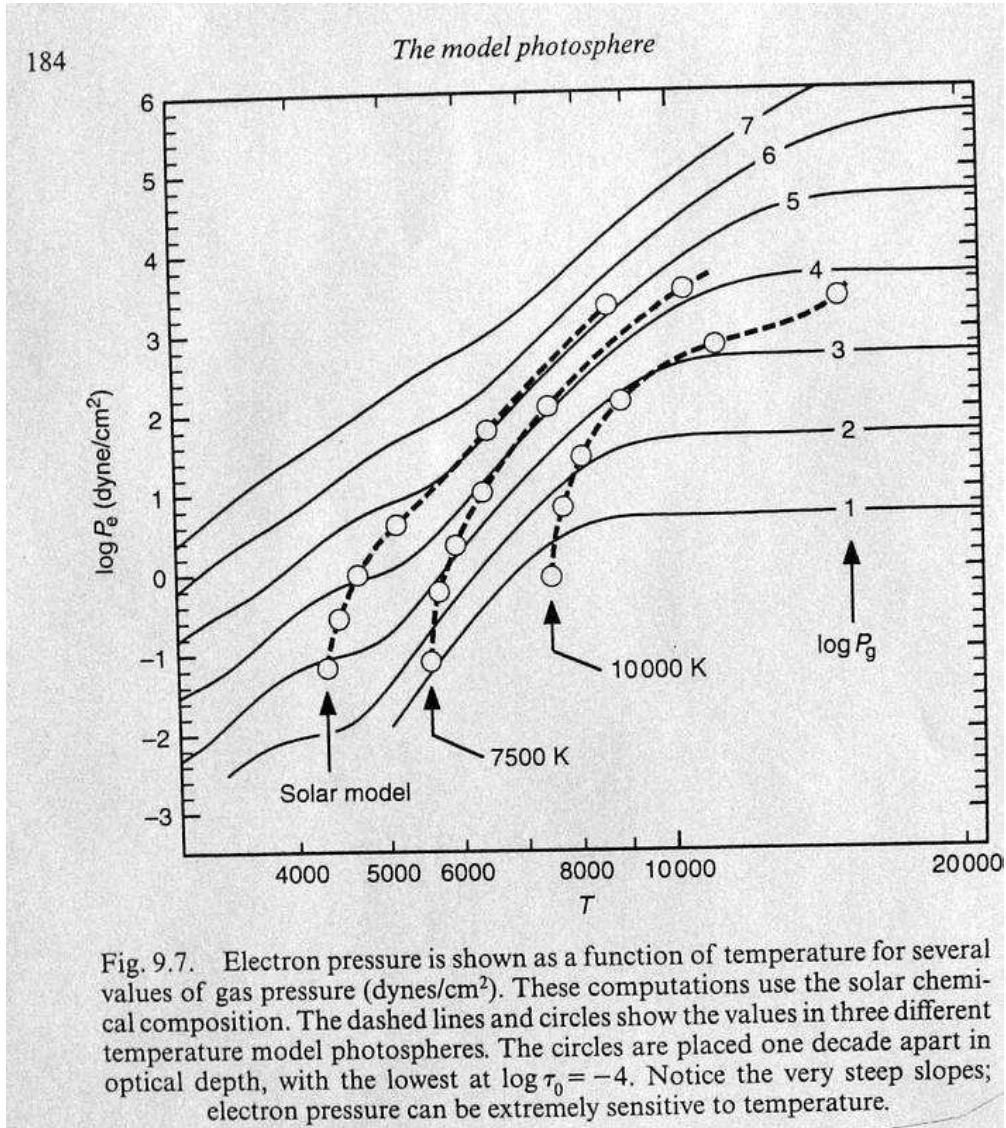


Fig. 1.— The electron pressure is shown as a function of T for several values of P_g for values appropriate to a stellar atmosphere. (Fig. 9.7 of Gray, 3rd edition)