Ay123

Fall 2007

STELLAR STRUCTURE AND EVOLUTION

Problem Set 2

Solutions due Wednesday, October 17th 2007

1. (5 pts) By considering a beam of particles with momentum p and velocity v, show how the pressure P is governed by the pressure integral equation quoted in class. By solving this integral for a beam of photons, derive the expression $P = \frac{1}{3}\, a\, T^4$ for the radiation pressure.

2. (10 pts) The internal structure of a low mass hydrogen-burning star can be approximated by a polytrope of index γ=5/3.

   (a): Assuming the stellar composition of the star is 75% hydrogen by mass and 25% helium and that the material is fully ionized, calculate the mean molecular weight μ.
   (b): If gas pressure dominates, show that the central temperature is given by an expression of the form $T_c \propto \mu M \,/\, R$. Calculate the constant of proportionality and hence establish an absolute scale using material given in the lectures.
   (c): For low mass stars, it is observed that stars of mass M have radii R that scale $\propto M^{0.08}$. Use the polytropic relations to calculate the run of the central pressure P_c and temperature T_c with mass.
   (d): For stars of what mass will the central radiation pressure, estimated from Q1, become equal to P_c?

3. (5pts) The equation of state for an isothermal gas sphere satisfies the relation $P \propto \rho$ which is equivalent to a polytrope with γ=1. Derive the associated value of n. Using the equation of hydrostatic equilibrium, show that if $\rho = \rho_c\, e^{-\psi}$ the resulting Lane-Emden equation is:

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2 \frac{d\psi}{d\xi}) = e^{-\psi}$$

   By considering the boundary conditions, explain why no star can be usefully represented by an isothermal gas sphere. What do you think might be the likely applications in astrophysics for such an equation of state?

4. (10 pts) Show that when a self-gravitating body of polytropic gas shrinks homologously and adiabatically, its thermal energy scales with radius R as $E_{\rm th} \propto R^{3(1-\gamma)}$. Hence, show that a polytropic star is unstable to gravitational collapse if $\gamma < 4/3$.

5. (10pts) The material in the envelope of a star is an ideal gas with γ=4/3 and the star is sufficiently centrally concentrated that the mass in the envelope is negligible compared to that in the core, M. If the envelope is just marginally convectively unstable, show that the temperature within the envelope varies with radius as

$$T = \frac{GM\mu m_H}{4k}(\frac{1}{r} - \frac{1}{r_s}) + T_s$$

where r_s and T_s refer to surface values.

6. (10pts) The aim is to calculate the Rosseland mean opacity in the case of free-free absorption in pure hydrogen. The frequency-dependent opacity is given by the expression:

$$\kappa_\nu \rho = 1.32\,10^{56} \frac{\rho^2 g_{ff}}{\nu^3 T^{1/2}} \,(1\;-\;e^{-h\nu/kT})\; cm^{-1}$$

where g_{ff} is a constant quantum mechanical correction factor called the *Gaunt factor*.

(i) First derive an expression for $\partial\, B_\nu / \partial\, T$

(ii) Next, introduce a dimensionless variable $x = h\nu\,/\,kT$

(iii) Derive an expression for $\frac{1}{\rho\kappa_\nu}\,\partial\, B_\nu/\partial\, T$ and plot the resulting function. Use the plot to argue that the Rosseland mean opacity is largely determined by κ_ν when the frequency ν is a few times kT/h.

(iv) Hence show that the Rosseland mean opacity for free-free absorption obeys Kramer's law where $\kappa \propto \rho T^{-3.5}$.

7. (10 pts) In this problem we will consider opacity due to the negative hydrogen ion, H^- in stellar atmospheres. In cool stars, the bound-free contribution from this ion dominates the opacity above the Balmer limit (λ=3647 Å), below which the opacity is predominantly due to photoionization of hydrogen atoms in the n=2 level. The ionization energy of H^- is χ^-=0.754 eV, meaning that this ion can absorb photons with wavelengths λ <16,444 Å

a: First, familiarize yourself with the Saha equation, which governs the number densities in various degrees of ionization, and the Boltzmann expression for number densities in various excited states (e.g. HKT pp152-159).

b: Write an expression for $n(H^-)/N(H)$ in terms of χ^-, T and the electron pressure $P_e = n_e\, kT$.

b: Calculate the ratio $n(H; n = 2)/n(H; n = 1)$ of the abundance of hydrogen atoms in the first excited state (n=2) to that in the ground state (n=1). Do the same for $n(H; n = 3)/n(H; n = 1)$.

c: Assuming an electron pressure, $P_e = 10^{1.5}$ dynes cm^{-2}, at what temperature does $n(H : n = 3) \simeq n(H^-)$? Assuming the bound-free absorption coefficients for H^- and the n=3 level of hydrogen are comparable, this gives the temperature below which H^- absorption dominates the opacity above the Balmer break.

d: Write an expression for κ(3647+)/κ(3647-), the ratio of opacities just above and below the Balmer break for (i) low temperature cases where the opacity above is due to H^- and (ii) high temperature cases where the opacity above is due to bound-free absorption from n=3 level of neutral hydrogen. Your results should show that at low temperatures, the Balmer discontinuity depends on both T and P_e, while at high temperatures it depends only on T.